

Classical paradoxes of locality and their possible quantum resolutions in deformed special relativity

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Abstract

In deformed or doubly special relativity (DSR) the action of the Lorentz group on momentum eigenstates is deformed to preserve a maximal momentum or minimal length, supposed equal to the Planck length, $l_p = \sqrt{\hbar G}$. The classical and quantum dynamics of a particle propagating in κ -Minkowski spacetime is discussed in order to examine an apparent paradox of locality which arises in the classical dynamics. This is due to the fact that the Lorentz transformations of spacetime positions of particles depend on their energies, so whether or not a local event, defined by the coincidence of two or more particles, takes place appears to depend on the frame of reference of the observer. Here we discuss two issues which may contribute to the resolution of these apparent paradoxes. First it may be that the paradox arises only in the classical picture, because it is inconsistent to study physics in which $\hbar = 0$ but $l_p = \sqrt{\hbar G} \neq 0$. Second, there may still be an observer independent notion of a local interaction, which slightly extends the usual notion without coming into conflict with the observed locality of interactions in nature.

These considerations may be relevant for phenomenology such as observations by the Fermi observatory, because at leading order in $l_p \times \text{distance}$ there is both a direct and a stochastic dependence of arrival time on energy, due to an additional spreading of wavepackets.

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1 Introduction

Doubly or deformed special relativity (DSR) is the hypothesis that the Poincaré group or its action is deformed to take into account the possibility of a maximal momentum or energy for individual elementary particles, without violating the relativity of inertial frames[1, 2, 3]. While this is an attractive idea, not least because it is accessible to investigation by current experiments[4], interpretations of and predictions for these experiments have been challenged by several confusions as to the interpretation of *DSR* in spacetime[5, 6]. These involve the notions of locality and velocity. The purpose of this paper is to propose an origin for these confusions which afflict spacetime descriptions of *DSR* and to investigate three features of *DSR* theories which may play a role in resolving them.

One very physical way to understanding the idea of *DSR* is to see it as a phenomenological description which arises in a particular limit of some underlying quantum theory of gravity¹. In this limit we take,

$$\hbar \rightarrow 0, \text{ AND } G \rightarrow 0 \quad (1)$$

in such a way that the dimensional constant²

$$M_p = \sqrt{\frac{\hbar}{G}} \rightarrow \text{constant} \quad (2)$$

is held fixed. (We work here in units where $c = 1$, since we assume there is still an invariant velocity.) That is, we turn off both quantum mechanics and gravity, but in such a way as to preserve phenomena depending on M_p and c . *DSR* is then something like the smile that is left of the Cheshire cat of quantum gravity. We can call this the classical *DSR* limit of quantum gravity. Just like the better studied limits in which we take $\hbar \rightarrow 0$ or $G \rightarrow 0$ separately, this is a limit that must exist if the quantum theory of gravity is well defined.

It is easy to state how physics may be modified in the *DSR* limit: momentum space becomes curved, with the radius of curvature measured by the invariant M_p . There are then two cases to discuss depending on how momentum space may be curved. Poincaré invariance may be broken, in which case the symmetry group of momentum space will have fewer than the ten generators of the Poincaré algebra. Or, if we want to preserve the existence of a ten parameter symmetry group, then momentum space must have constant curvature, ie it has a deSitter or anti-deSitter geometry. This results in a deformation of Poincaré invariance. This is the basic idea of *DSR*.

So long as we stay in momentum space the implementation of this idea is straightforward, but issues develop when we ask for the effect on physics in the complementary spacetime description. These problems can be seen to arise because we are working in a limit in which \hbar has been taken to zero. In this case there is no fixed length or time corresponding to the mass fixed in (2). Instead,

$$t_p = \sqrt{\hbar G} \rightarrow 0 \quad (3)$$

This means that the classical *DSR* limit only yields something new in momentum space. When applied to physics in spacetime, the classical *DSR* limit is ordinary special relativity.

Another way to say this is that the relationship between momentum space and spacetime depends on \hbar being non-zero. We need it to make sense of the fourier transform, without it we

¹This viewpoint was first proposed by Kowalski-Glikman[3]

²In $d = 3$, the number of spatial dimensions.

could not write $e^{i\frac{x^a p_a}{\hbar}}$. Alternatively when \hbar is zero x^a and p_a commute, so the idea that one generates translations in the other disappears. Hence, *DSR* can be only understood as a classical theory in momentum space. If we want to translate the physics of *DSR* into spacetime we need $\hbar \neq 0$ which means we must work with the quantum dynamics.

In spite of this, there have been attempts to describe *DSR* physics in the language of classical dynamics of spacetime. These have given rise to some confusion. Indeed, as pointed out by Schtzhold and Unruh[5] and Hossenfelder[6] there are apparent paradoxes that challenge the consistency of the description. They describe thought experiments in which the simple question of whether the world lines of three or more particles coincide at a particular local event in spacetime appears to be observer dependent. This challenges the basic operational definition of a spacetime event proposed by Einstein, according to which an event is defined by the coincide of several particles in space and time.

From the point of view just mentioned, it is not surprising if problems appear when one attempts to discuss the physics of *DSR* in terms of classical physics in spacetime, because quantum effects are being treated inconsistently. If we include effects coming from a finite $l_p = \sqrt{\hbar G}$ we must include quantum effects because \hbar is non-zero. This suggests an hypothesis about the paradoxes raised in [5] and [6], which is that they perhaps arise because it is inconsistent to reason about *DSR* effects in spacetime as if quantum mechanics was turned off, but l_p is still non-zero, because we are being inconsistent about the dependence of observable quantities in \hbar .

1.1 The strategy of this paper

It is one thing to suggest an hypothesis about a problem and another to show that it cleanly solves the problem. When seeking to go further, however, we run into a problem, which is that the scenarios described in [5] and [6] are discussed in somewhat heuristic contexts. This is sufficient to convince one there are issues worth worrying about, but may be insufficient to resolve them, because it is not clear which precise physical theories-if any- correspond to the assumptions that are made there. For better or worse there are several distinct formulations of theories which are motivated by the idea of *DSR*, and it is not known if they are equivalent; nor is it known in all cases if these formulations are completely self-consistent. There are also two distinct issues that are easily confused in these discussions: 1) Is a particular formulation of *DSR* internally consistent? 2) If it is consistent, does it lead to predictions that disagree with well confirmed observations such as the locality of physics?

Because of these issues we follow a cautious strategy in addressing the apparent paradoxes raised in [5] and [6]:

1. We work within the best defined formulation of *DSR*, which is physics on the non-commutative manifold κ -Minkowski spacetime. For simplicity we discuss in this paper only the theory of free relativistic particles on κ -Minkowski spacetime[7, 8, 9, 10, 11].
2. In this context we formulate a paradox similar to those discussed by [5] and [6].
3. We discuss in this paper two approaches to resolving that paradox, one having to do with additional quantum effects special to *DSR* theories, a second having to do with a possible relaxation of the notion of locality.

4. A third strategy of resolving the paradoxes is discussed in another paper, which is that the apparent non-localities are a coordinate artifact associated with an ambiguity in extending Einstein's procedure for synchronizing clocks to clocks with a finite frequency far from the origin of coordinates of a reference frame[25].

The results are tentative, in that we find evidence that both effects may play a role, but we are not able to definitively show the problem is solved. It must also be emphasized that because of the strategy we choose we cannot address directly the paradoxes of Schützhold and Unruh[5] and Hossenfelder[6]³. We can here only address similar issues which appear in κ -Minkowski spacetime; whether the same insights apply to the cases discussed in [5, 6] can only be resolved by further work.

1.2 Outline of the argument

This having been said, let us introduce the basic ideas and results that are discussed below.

It is easy to see how an issue with locality arises in κ -Minkowski spacetime. The details are in the sections below but for the sake of clarity of argument we can now sketch the key points of the argument.

If momentum space is curved then translations on momentum space don't commute with each other. But if the spacetime coordinates are constructed to be complementary to momentum space in the usual way, they are the generators of translations on momentum space, which means that they don't commute. So we have a non-commutative spacetime geometry. We have then to investigate whether this non-commutative spacetime can support a consistent framework for physics, which agrees with experiment.

The new commutation relations have to be consistent with a ten parameter algebra of symmetries, inherited from the symmetries of the deSitter geometry of momentum space. It turns out this can be achieved if we take it that the space coordinates do not commute with the time coordinate

$$[t, x^i] = t_p x^i \quad (4)$$

where $t_p = \sqrt{\hbar G}$ is the Planck time. This certainly gives rise to some general problems of interpretation because the notion of an event, which requires localizing two or more particles at the same point of space and time, appears to be compromised by the inability to make both space and time measurements simultaneously sharp⁴.

The next step is that, in order to preserve the commutation relations (4), the Lorentz transformations of the position and time coordinates of a relativistic particle turn out to depend on its energy and momentum[3]. (These transformations are reviewed in (37,38) below.)

This energy and momentum dependence of the Lorentz transformations leads directly to problems with locality. To see why, consider a scattering event defined by the coincidence of four worldlines of particles of different energy and momentum—two worldlines for the incoming particles and two worldlines for the outgoing particles. Suppose that one inertial observer sees them

³It should also be mentioned that Hossenfelder[6] does consider the possibility that quantum effects resolve the problem, but comes to a different conclusion than we do here. Whether that is because the models are different or for other reasons remains unclear.

⁴Alternative formulations of *DSR* which involve instead of (4) an energy dependent metric also give rise to similar confusions, but these will not be discussed here.

as coinciding at a single value of (x^i, t) . Suppose also we want to use a Lorentz transformation to derive the trajectories of those particles as seen by a second inertial observer. Then it is easy to construct cases in which the four particles no longer coincide in the second frame of reference, because the modified Lorentz transformation takes the positions coincident in one frame to different locations, depending on the energy and momentum of the particles.

This certainly sounds bad, as it means an interaction that looks local to one observer involves four separated events in another observer's description. However, we have argued that the classical picture of *DSR* may not be self-consistent, so this puzzle should be re-examined in the quantum theory. To see why quantum effects may help to resolve these apparent paradoxes, note that the commutation relation (4) imply additional uncertainty relations[16, 17, 18, 19],

$$\Delta t \Delta x \geq t_p |x| \quad (5)$$

This implies immediately that we cannot construct a quantum state in which we can precisely localize a single particle both in x^i and t . So the first observer cannot actually be sure that the four worldlines coincided at a single event. All observers must describe the possible interactions amongst the particles in terms of quantum probability amplitudes. Indeed, as is discussed by several authors[16, 17, 18, 19, 6, 23], this additional uncertainty gives rise to an anomalous spreading of wavepackets due to the modified commutations relations (5). We reproduce this below, and investigate the extent to which this may provide a solution to the problem of non-locality generated by energy and momentum dependent Lorentz transformations.

However, we are not able to demonstrate that the spreading of the wavepacket is sufficient to hide the non-locality generated by the Lorentz transformation for all states. Thus, we next investigate another approach to the issue. This is whether there might be a relaxation of the notion of locality which can incorporate the Lorentz dependence of the notion of a localized event, without at the same time leading to non-locality being so generic that it blatantly disagrees with known physics. Thus, in section 5 we introduce a notion of κ -locality in the context of the classical free relativistic particles, which is defined as follows: *A set of $N \geq 3$ events, E_i^a , each on a worldline x_i^a of a free relativistic particle, are κ -local if there is an (energy and momentum dependent) Lorentz transformation which takes the N events to a single event $E^{a'}$.* We can then hypothesize that interactions among particles propagating in κ -Minkowski spacetime are κ -local rather than local⁵. We can call $N \geq 3$ worldlines that contain mutually κ -local events, κ -intersecting.

This will be acceptable only if generic sets of $N \geq 3$ worldlines do not contain any mutually κ -local events. Equivalently, it should not be the case that $N \geq 3$ sets of worldlines can be brought to intersect by an energy and momentum dependent Lorentz transformation. For were this possible, the principle that physics is κ -local would imply that any set of $N \geq 3$ particles could be interacting. Were this the case the notion that physics is local would be entirely lost.

Happily, we show that this catastrophe does not occur. In section 5 we find that κ -locality is not a generic property of any set of three or more worldlines. To the contrary, the sets of κ -intersecting triples of worldlines are of measure zero in the sets of three worldlines, and become even rarer as four or more particles are involved. To investigate whether physics was κ -local rather than

⁵Note that we restrict consideration to 3 or more particles, because if there is a real interaction amongst two particles, either they exchange energy and momentum, in which case the outgoing worldlines have different energy and momentum than the incoming ones, or they annihilate into a third particle. Coincidences of two particles that do not change either's energy and momentum are not physical interactions.

local would then take very delicate experiments, with finely tuned initial and final conditions to restrict to the case where the worldlines of the ingoing and outgoing particles are all mutually κ -intersecting. Hence, the world could be κ -local rather than local, and we would not yet have noticed the distinction⁶.

There is a third point which may play a role in resolving the paradoxes of locality. Let us return to the time coordinate t . It is possible to choose the phase space description so that t commutes with the components of spatial momentum. Hence t can be used to discuss the dynamics in momentum space. However, as we shall see in the next section, this comes with a cost, which is that the usual position-momentum commutators are deformed to

$$[x^i, p_j] = i\hbar\delta_j^i f(p_0) \quad (6)$$

where $f(p_0)$ is the function

$$f(E) = e^{t_p p_0/\hbar} \quad (7)$$

This has an interesting consequence, which is that we can introduce a new time coordinate that does commute with the x^i . This is⁷

$$T = t + \frac{t_p}{\hbar f(p_0)} x^i p_i \quad (8)$$

It is easy to confirm that,

$$[T, x^i] = 0 \quad (9)$$

Hence, it is unproblematic to discuss events defined in a spacetime coordinatized by (x^i, T) , even in the quantum theory. *So it is natural to propose that the classical spacetime in which we observers make measurements is defined by clocks that measure T and rulers that measure x^i .*

However, T no longer commutes with p_i , instead

$$[T, p_i] = t_p p_i \quad (10)$$

Hence there is a good time coordinate, T , to discuss evolution of wavefunctions in position space. And there is a good time coordinate, t , to discuss evolution in momentum space. But they are related to each other by the non-local transform (8). This plays a role in the derivation of the spreading of wavepackets in (x^i, T) space.

While the aim of this paper is to address the issue of physical consistency of DSR , we note that there are phenomenological implications of the results derived below. By studying the propagation of a wavepacket in (x^i, T) spacetime we are able to study the question of the energy dependence of the speed of massless particles. We find that there is a first order variation of the speed of light with energy. In addition, there is another first order effect, which is the new contribution to spreading of wavepackets. This gives a stochastic variation of arrival times of photons proportional to $T t_p \Delta E$, where T is the time traveled and ΔE is the uncertainty in energy of the

⁶We may note that this appears to disagree with the claim of Hossenfelder in [6] that in this language can be translated as the assertion that κ -local physics is ruled out by experiment. However, it should be stressed that whether this is due to her model of DSR physics being different or to another cause is not clear at this time. What we can assert is that no claim can be made that physics in κ -Minkowski spacetime is grossly in contradiction with the observed locality of physical interactions.

⁷This is also discussed in [11].

wavepacket. This is at the same order as the linear dependence of velocity with energy, and so might be observable in current observations by Fermi and other observatories. A new strategy to bound or measure this kind of stochastic effect in the Fermi data needs to be developed.

The remainder of this paper is organized as follows. In the next section we describe a general approach to deforming the quantum physics of a free particle in Minkowski spacetime and then show how it can be specialized to κ -Minkowski spacetime. In section 3 we show how paradoxes of locality can be generated by studying inconsistently classical physics in κ -Minkowski spacetime. In section 4 we investigate the extent to which these apparent paradoxes of locality may be resolved in the quantum theory of a free relativistic particle⁸. Then in section 5 we introduce the concept of κ -locality and κ -intersecting and show that they are very non-generic properties of sets of worldlines. We conclude by listing some of the open issues that remain to be resolved before it can be asserted that *DSR* is either can or cannot be fully realized within quantum physics and hence before its experimental implications can be unambiguously predicted.

2 *DSR* for a free particle in terms of non-commutative geometry

2.1 An algebraic approach to *DSR*

We begin by examining how the Hilbert space for a single free relativistic particle can be deformed consistently. We start with a set of possible deformed commutation relations.

$$[x^i, p_j] = i\hbar\delta_j^i f(p_0), \quad [t, p_0] = i\hbar g(p_0) \quad (11)$$

$$[t, x^i] = i t_P x^i h(p_0) \quad (12)$$

with the rest vanishing. In particular, we assume,

$$[t, p_i] = 0 \quad (13)$$

because I would like to define the quantum evolution in a time that commutes with momentum so I can evolve states on momentum space in the usual way.

By checking the Jacobi relation

$$0 = [t, [x^i, p_j]] + \dots \quad (14)$$

we find that

$$\frac{f'}{f} = \frac{t_P}{\hbar} \frac{h}{g} \quad (15)$$

It is important to note that the nonvanishing of (12) follows from the deformations in (11) by the Jacobi relations. So a non-commutativity of space and time coordinates is a natural consequence of the deformation of the canonical commutation relations for a relativistic particle. What this means is that we cannot speak of events or evolve position space wavefunctions in the usual way, so long as we use the time coordinate t .

⁸We note that the uncertainty relations and the resulting spreading of wavepackets have been discussed early in the literature on κ Minkowski spacetime and *DSR*[16, 17, 18, 19]. What is new here is only the suggestion that these may be necessary to resolve the apparent paradoxes of locality arising from the dependence of boosts of spacetime coordinates on energy.

2.2 Review of κ -Poincaré

Let us first briefly review physics in κ -Minkowski spacetime[7, 8, 9, 10, 11, 3]. The basic idea is that momentum space is a deSitter spacetime coordinatized by k_0 and k_i with a metric

$$ds^2 = -dk_0^2 + e^{\frac{2k_0}{E_p}} dk_i dk_i \quad (16)$$

The commutation relations are

$$[x^i, k_j] = i\hbar\delta_j^i, \quad [t, k_0] = i\hbar \quad (17)$$

$$[t, x^i] = it_p x^i \quad (18)$$

$$[t, k_i] = -it_p k_i \quad (19)$$

In particular, note that unlike what we have assumed above, the commutator of t with k_i is non-zero.

The dynamics is defined by a Hamiltonian constraint constructed from the Casimir of the κ -Poincaré algebra. This is the invariant length on the curved momentum space, which is invariant under an $SO(1, 3)$ subgroup of the deSitter group.

$$\mathcal{H} = 4E_p^2 \sinh^2\left(\frac{k_0}{2E_p}\right) - k_i k_i e^{\frac{k_0}{E_p}} - m^2 = 0 \quad (20)$$

An integration measure on momentum space, invariant under the non-linear action of the Lorentz group is defined by

$$d\omega = dk_0 \wedge d^3k e^{\frac{3k_0}{E_p}} \quad (21)$$

2.3 Correspondence with κ -Poincaré

To construct the quantum theory we prefer to work with our original ansatz according to which $[t, p_i] = 0$ (13). This way we can evolve the eigenstates of momentum p_i in the time t . We can see that this corresponds to the solution of the Jacobi relations given by

$$f = e^{\frac{k_0}{E_p}}, \quad h = g = 1 \quad (22)$$

with the relation

$$p_i = e^{\frac{k_0}{E_p}} k_i, \quad p_0 = k_0 \quad (23)$$

The commutation relations are then

$$[x^i, p_j] = i\hbar\delta_j^i e^{\frac{p_0}{E_p}}, \quad [t, p_0] = i\hbar \quad (24)$$

$$[t, x^i] = it_p x^i \quad (25)$$

$$[t, p_i] = 0 \quad (26)$$

In terms of these variables the metric on the de Sitter momentum space is

$$ds^2 = -dp_0^2 \left(1 - \frac{p_i p_i}{E_p^2}\right) - \frac{p_i dp_i dp_0}{E_p} + dp_i dp_i \quad (27)$$

The Casimir of κ -Poincaré now takes the form,

$$\mathcal{H} = 4E_p^2 \sinh^2 \left(\frac{p_0}{2E_p} \right) - p_i p_i e^{-\frac{p_0}{E_p}} - m^2 = 0 \quad (28)$$

and the invariant measure is

$$d\omega = dp_0 \wedge d^3p \quad (29)$$

Because of (25) we cannot discuss evolving the position of the particle in the time t . However, because of (26), the time t is suitable for evolving the particle in momentum space.

To evolve the system in the position coordinates x^i we define a new time variable (8) which obeys

$$[T, x^i] = 0 \quad (30)$$

However,

$$[T, p_i] = it_P p_i \quad (31)$$

so T cannot be used to evolve wavefunctions in momentum space parameterized by p_i .

Thus, if we take p_i for the spatial momenta, we see the very interesting conclusion that the evolution in position space and momentum space must take place with different time variables, whose relation to each other, as defined by (8), is non-local in both position and momentum space.

We can however eliminate (31) by going back to measuring spatial momenta in terms of k_i , because we have,

$$[T, k_i] = 0 \quad (32)$$

3 Classical paradoxes of locality and their quantum resolutions

We now show how paradoxes of locality can be generated by inconsistently taking $\hbar = 0$ but $t_P \neq 0$. Then we show how they may be resolved when \hbar is turned back on.

3.1 Non-locality

We can see just from the algebra of observables that there will be apparent issues with non-locality if we use the wrong set of coordinates. Suppose that we subject our particle to sudden force coming which is local in space so it occurs at a particular $x^i = a^i$. Since x^i and T commute we can localize the event precisely also in T , so that it occurs at a particular $T = T_a$. Thus, in the (x^i, T) variables, the force can be modeled as coming from from a potential

$$V(x, T) = \delta^3(x^i - a^i) \delta(T - T_a) \quad (33)$$

Note that we could not write a potential local in terms of x^i and t because they don't commute. Hence,

$$V'(x, t) = ? \delta^3(x^i - a^i) \delta(t - t_a) \quad (34)$$

is undefined as it is beset with operator ordering issues.

So let us stick with the first event, local in x^i and T , but suppose we want to describe when it happens in terms of the other time variable, t . We will have that it takes place at a^i but at a different, momentum dependent time, given by

$$t = T_a - \frac{t_P}{\hbar f(p_0)} a^i p_i \quad (35)$$

however, we cannot measure T and p_i at the same time, since $[T, p_i]$ doesn't vanish. Alternatively, since x^i is sharply defined, p_i is maximally uncertain. So we cannot predict when the event will take place in the t coordinate.

Conversely, if we measure p_i we can determine that the event takes place at a definite $t - t_a$ with a particular momentum b_i . But then in terms of the spacetime variables, the event will take place at a position dependent time

$$T = t + \frac{t_P}{\hbar f} x^i b_i \quad (36)$$

but since p_i has been measured sharply, x^i will be maximally uncertain, so when the event takes place in terms of T will be maximally uncertain.

These facts have to be taken into account carefully in any description of events in spacetime. To investigate their effect on propagation of particles we will in the next section construct the quantum theory of a single relativistic particle. But first we see how the paradoxes we referred to in the introduction arise.

3.2 Boosts and events

The issues that give rise to the apparent paradoxes of Unruh et al and Hossenfelder become apparent when one writes down the Lorentz transformations for position in κ -Minkowski spacetime. From [12, 13] we find for a boost of magnitude $\beta\gamma$ denoted by a spatial vector ω^i , the position and time coordinates transform as

$$\delta x^i = -\omega^i t - t_P \epsilon^{ijk} \omega_j L_k \quad (37)$$

$$\delta t = -\omega \cdot x + t_P \omega \cdot N \quad (38)$$

where L_i are the spatial angular momentum generators

$$L^i = \frac{1}{f} \epsilon^{ijk} x_j p_k \quad (39)$$

and N_i are the generators of deformed boost transformations

$$N_i = -p_i e^{-\frac{p_0}{E_p}} t - x_i \left[\frac{E_p}{2} (1 - e^{-\frac{2p_0}{E_p}}) + \frac{1}{2E_p} p \cdot p e^{-\frac{2p_0}{E_p}} \right] \quad (40)$$

One can also check that

$$\delta T = -\omega \cdot x (1 + t_P r(p_0)) + t_P \omega \cdot p e^{-\frac{p_0}{E_p}} \left(T - \frac{t_P}{\hbar} x^i p_i e^{-\frac{p_0}{E_p}} \right) \quad (41)$$

where

$$r(p_0) = \frac{1}{2t_P} (1 - e^{-2t_P p_0}) + t_P p \cdot p e^{-2t_P p_0} \approx p_0 + \dots \quad (42)$$

To first order in t_p we have

$$\delta T = -\omega \cdot x(1 + t_p p_0) + t_p \omega \cdot p T + O(t_p^2) \quad (43)$$

We now see how some apparent paradoxes stem from these transformation laws.

3.3 Transverse length contraction and relativity

We note first of all, that from (37), directions perpendicular to the direction of the boost can contract. This gives rise to an apparent paradox. Consider two inertial observers, Alice and Bob, who are approaching each other along their \hat{z} axes, each of whom carries a stick along their \hat{x} axis. Suppose that by (37) Alice sees Bob's stick contract. What does Bob see? By the relativity of inertial frames Bob should also see Alice's stick contract. This is what we say with sticks parallel to their relative velocities and this turns out to be consistent. But now let us note that as they pass Alice and Bob can mark where the end of the other's stick passes their stick. This does not yield a problem when they are held parallel because of the relativity of simultaneity but it is a problem now, because the events of marking the ends of the sticks are simultaneous in both observer's frames. Hence, if Alice sees Bob's stick to have contracted relative to hers, Bob must agree that Alice's stick is longer. But this appears to violate the relativity of inertial frames. This is why ordinarily we do not have contraction of directions perpendicular to motion in special relativity.

The resolution of this problem is that the contraction of the perpendicular directions is proportional to conserved quantities, which in this case is the \hat{y} component of angular momentum of the sticks. So whether it is Alice or Bob who sees the other's stick as shorter than theirs depends on which stick has a larger \hat{y} component of angular momentum.

3.4 A classical locality paradox

There are other more serious apparent paradoxes connected with the fact that the transformations (37) and (38) are dependent on energy, momentum and angular momentum of the objects which are being transformed.

Here is a prototype of an apparent paradox involving transformations between the observations of two inertial observers Alice and Bob.

Suppose that Alice sees a collision of two particles at a position a^i and time $T = s$ in her frame. We can use the transformations (37) and (38) to compute the first order positions and times of the particles at the collision, as they will be seen according to Bob's instruments. Bob will see the two particles to have positions and times given by

$$x^{i'} = a^i + \delta x^i, \quad T' = s + \delta T \quad (44)$$

where δx^i and δT are given by (37) and (41), respectively.

If the two particles have different values of energy, momentum and angular momentum, Bob will see the event that Alice sees as two particles colliding as corresponding to two events, separated in space by a vector with space and time components Dx^i and DT , given by

$$Dx^i = t_p \left(\omega^i a^k Dp_k - \epsilon^{ijk} \omega_j DL_k \right) \quad (45)$$

$$DT = -t_p (\omega \cdot a DE - s \omega \cdot \Delta p) \quad (46)$$

where Dp_k, DE and DL^i are the differences in conserved quantities carried by the two particles, as observed by Alice, ie $Dp^k = p_1^k - p_2^k$, etc.

Thus, the two particles that Alice sees coincide need not be seen to coincide by Bob. Indeed, since there is a shift perpendicular to the direction of the boost it is possible that they never collide at all. It is thus easy to construct apparent paradoxes by, for example, supposing that the two particles have a short ranged interaction that scatters them. Suppose that the boost to Bob's frame is large enough that the Dx are larger than the interaction range. Does Alice see them to scatter, but Bob not?

4 Quantum theory of a free relativistic particle

The purpose of this section is to investigate whether quantum dynamics can contribute to the resolution of the puzzle we have just discussed.

We study the quantum dynamics of a free particle in κ -Minkowski spacetime. and show that there is an anomalous spreading of the wavepacket proportional to $t_p \times \text{distance}$. This shows that quantum effects could contribute to the resolution of the locality paradox.

4.1 Bases in Hilbert space

We begin our study of quantum physics in κ -Minkowski spacetime by constructing the Hilbert space, \mathcal{H} , by constructing eigenstates of complete commuting sets of operators.

We will see shortly that a key point of our approach is that the dynamics is defined first in momentum space, then transformed to the commutative spacetime defined by the (x^i, T) operators.

One complete commuting set of observables is composed of (E, p_i) . They define a basis

$$\hat{p}_i |E, p\rangle = p_i |E, p\rangle, \quad \hat{E} |E, p\rangle = E |E, p\rangle \quad (47)$$

with completeness relation given by the measure (29)

$$1 = \int dE d^3p |E, p\rangle \langle E, p| \quad (48)$$

Another complete commuting set is (t, p_i) . These are useful for discussing evolution in time in momentum space. These define a basis

$$\hat{p}_i |t, p\rangle = p_i |t, p\rangle, \quad \hat{t} |t, p\rangle = t |t, p\rangle \quad (49)$$

with completeness relation

$$1 = \int dt d^3p |t, p\rangle \langle t, p| \quad (50)$$

We note that because of $[t, E] = i\hbar$ we have

$$\langle t, p | E, p' \rangle = \delta^3(p, p') e^{iEt/\hbar} \quad (51)$$

We note that there is no basis which simultaneously diagonalizes t and x^i .

If we want to discuss evolution in position space we have to use a different time coordinate, T , which is part of another complete set of commuting observables, given by (T, x^i) . They define a basis by

$$\hat{x}^i|T, x \rangle = x^i|T, x \rangle, \quad \hat{T}|T, x \rangle = T|T, x \rangle \quad (52)$$

with completeness relation

$$1 = \int dT d^3x |T, x \rangle \langle T, x| \quad (53)$$

To transform from an energy basis to evolution in position space then requires two steps. First we have to change from energy to the time t by using the relation (51).

Next we change from the (t, p) basis to the (T, x) basis through amplitudes

$$\langle t, p|T, x \rangle = e^{i \frac{x^i p_i}{T(E)\hbar}} \delta(T - t - \frac{t_P}{\hbar f} x^i p_i) \quad (54)$$

This gives us the non-local transform

$$\begin{aligned} \Psi(x^i, T) &= \langle T, x|\Psi \rangle \\ &= \int d^3p dt d^3p' dE \langle T, x|t, p \rangle \langle t, p|E, p' \rangle \langle E, p'|\Psi \rangle \\ &= \int d^3p dE e^{-i \frac{x^i p_i}{T(E)\hbar} + iE(T - \frac{t_P}{\hbar f} x^i p_i)} \Psi(E, p) \end{aligned} \quad (55)$$

4.2 Dynamics in momentum space

We will impose dynamics by defining a modified dispersion relation in energy-momentum space. We define the dynamical Hamiltonian constraint operator by the quantization of (28).

$$\hat{\mathcal{H}} = 4E_p^2 \sinh^2 \left(\frac{E}{2E_p} \right) - p_i p_i e^{-\frac{E}{E_p}} - m^2 = 0 \quad (56)$$

We define the physical Hilbert space \mathcal{H}_{Phys} to be the subspace of \mathcal{H} defined by

$$\hat{\mathcal{H}}|\Psi \rangle = 0 \quad (57)$$

In energy-momentum space the solutions of this are given by

$$\Psi(E, p) = \delta(4E_p^2 \sinh^2 \left(\frac{E}{2E_p} \right) - p_i p_i e^{-\frac{E}{E_p}} - m^2) \chi(p). \quad (58)$$

4.3 Dynamics in position space

4.3.1 Propagation and spreading of a wavepacket

We now perform the transform (55) in the case of one spatial dimension with a solution (58) with

$$\chi(p) = e^{-\frac{(p-p_0)^2}{2\Delta p^2}} \quad (59)$$

With $\Delta x = \frac{f(E)}{\Delta p} \approx \frac{1}{\Delta p}$ the result is proportional to

$$\Psi(x, T) \approx \exp \left(-\frac{1}{2\Delta w^2} [(x - T)^2 - 8xl_p p_0(x - T)] + \frac{i}{1 + \frac{4x^2 l_p^2}{\Delta x^4}} [p_0(x - T) + \frac{2xl_p(x - T)^2}{\Delta x^4}] \right) \quad (60)$$

where there is a new width

$$\Delta w = \sqrt{\Delta x^2 + x^2 l_p^2 \Delta p^2} \quad (61)$$

We see that there is a new effect proportional to l_p in which the width spreads out as the particle propagates.

When the particle reaches an $x > \Delta x \frac{\Delta x}{l_p}$ the width grows as

$$\Delta w \approx xl_p \Delta p \quad (62)$$

This implies there is a stochastic effect on the time of propagation which is linear in l_p and of the same order as the linear variation in velocity.

It is interesting to ask whether this additional wave packet spreading can address the paradoxes of locality we discussed in section 3.4. We can be sure that the resulting quantum uncertainty would dominate over the locality paradox if in every frame of reference

$$\Delta w > Dx, \quad \Delta w > DT \quad (63)$$

Neglecting the transverse term proportional to the angular momentum we see that this would be the case if for all particles,

$$\Delta p > |p|. \quad (64)$$

This implies, using (45) and (46) that

$$\Delta w \approx xl_p \Delta p > xl_p p > |\omega| xl_p Dp = |Dx| \quad (65)$$

Therefore, for states where (64) is true in every frame of reference the quantum uncertainty will dominate over the apparent non-localities created by the momentum dependence of lorentz transformations. This is encouraging, but not definitive, for one thing because there are states for which (64) is not satisfied.

4.3.2 Wave equation in spacetime

We have constructed the wavefunction by transforming from a wavepacket in momentum space, but this should be equivalent to solving a wave equation in spacetime. The problem is that the corresponding wave equation, while linear in $\Psi(x, T)$ is a complicated function of space and time derivatives. To see what it is, we transform the constraint (56) to a wave-equation on spacetime. Using the transform (55) we find that

$$\hat{E} \rightarrow -i\hbar \frac{\partial}{\partial T}, \quad \hat{p}_i \rightarrow -i\hbar e^{-it_p \frac{\partial}{\partial T}} \frac{\partial}{\partial x^i} \quad (66)$$

So the wave equation gotten by transforming (56) is

$$\hat{H}\Psi(x^i, T) = \left(\frac{4}{t_p^2} \sinh^2(-i\hbar \frac{\partial}{\partial T}) - e^{it_p \frac{\partial}{\partial T}} \nabla^2 - m^2 \right) \Psi(x^i, T) = 0 \quad (67)$$

We see that the spacetime wave equation (67) is of infinite order in time derivatives. This means that while every transform of a solution to (56) on momentum space may solve (67), the latter may have many more solutions that do not arise from the transform of a solution on momentum space⁹. This is another reason to take the momentum space description as fundamental. We can take the point of view that only solutions to (67) that arise from a transform from momentum space be taken as physically acceptable solutions. This is a function on a three manifold's worth, which is correct.

4.4 Velocity in the commutative space-time coordinates (x^i, T)

There has been a lot of confusion as to the definition of velocity of particles in κ -Minkowski spacetime. We can see why, by working out the phase velocity as well as the classical point particle velocity from the canonical theory. We will see that they are not the same.

4.4.1 Phase velocity

It is straightforward to compute the phase velocity from the wavefunction for the wave packet or, equivalently, from the plane wave,

$$\Psi_p(x^i, T) = e^{-i[\frac{x^i p_i c - t_p E_p}{\hbar} - E_p(T - t_p \frac{x^i p_i c - t_p E_p}{\hbar})]} \quad (68)$$

where E_p is given by the solution of (28).

The principle of stationary phase gives, for propagation in the z direction

$$v = \frac{dz}{dT} = \frac{E_p}{p_z e^{-t_p E_p} (1 + t_p E_p)} = \frac{e^{\frac{t_p E_p}{2}}}{(1 + t_p E_p)} \approx 1 - \frac{t_p E}{2} + \dots \quad (69)$$

so we see there is a leading order subliminal dependence of the speed of light with energy.

4.4.2 Classical computation of velocity

It is also possible to define a classical notion of velocity using the evolution generated by the Hamiltonian constraint, \mathcal{H} . This follows the standard computation, ([14]) for $\frac{dx^i}{dt}$. However we argue that because x^i and t don't commute, it is more relevant for the classical limit of the quantum theory to compute the velocity in the commutative spacetime coordinates, $\frac{dx^i}{dT}$. This is defined by the ratio

$$\frac{dx^i}{dT} = \frac{dx^i}{d\tau} \left(\frac{dT}{d\tau} \right)^{-1} \quad (70)$$

where τ is the arbitrary time parameter which is used to parameterize evolution generated by the Hamiltonian constraint.

$$\frac{dx^i}{d\tau} = \{x^i, \mathcal{N}\mathcal{H}\} = -2\mathcal{N}p_i \quad (71)$$

where \mathcal{N} is an arbitrary lapse function. We find also that

$$\frac{dT}{d\tau} = \{T, \mathcal{N}\mathcal{H}\} = \mathcal{N} \left(\frac{\partial \mathcal{H}}{\partial E} + t_p p_i \frac{\partial \mathcal{H}}{\partial p_i} \right) = \mathcal{N} \left(4 \sinh\left(\frac{t_p E}{2}\right) \sinh\left(\frac{t_p E}{2}\right) - t_p^2 p^2 e^{-t_p E} \right) \quad (72)$$

⁹For a contrary view please see [26].

The result after using the Hamiltonian constraint is

$$\left| \frac{dx^i}{dT} \right| = e^{t_p} E \quad (73)$$

We note that this is different from the phase velocity computed above.

It also differs from the result [14]

$$\left| \frac{dx^i}{dt} \right| = 1 \quad (74)$$

so whether there is an energy dependent speed of light at the classical level does indeed depend on the definition of time used.

5 Redefining locality

It is clear to begin with that the usual principle of locality cannot be realized in a quantum theory which includes the commutation relations (4). What then must be investigated is whether the notion of locality can be relaxed sufficiently to incorporate (4) while remaining sufficiently restrictive that it is not ruled out by our existing store of knowledge. In this section we make such a proposal and argue that it is narrow enough to not yet have been ruled out by experiment.

In the following we restrict ourselves to the context in which the paradoxes of locality originally presents the paradox, which is the classical dynamics of worldlines of free relativistic particles. We will consider cases where three or more world-lines intersect at an event, because unless the outgoing states are different from the incoming states we cannot say that an interaction has taken place. We thus consider a class of events involving $N \geq 3$ particles in which one inertial observer, named Alice, sees all N worldlines to intersect at a single point, E^a . For example, two of them may be incoming states and two of them outgoing, in which case $N = 4$. Alice will then explain the fact that the particles scattered, exchanging energy and momentum, as caused by the two incoming particles's worldlines having intersected.

However, if *DSR* is in fact observer independent, then another inertial observer, Bob, must observe that the scattering take place as well. However, because of the energy and momentum dependence of the Lorentz transformations, Bob may see four worldlines that never intersect. Instead, the event E^a has been split into four events. Bob will be forced to describe the history Alice sees as one in which there are two incoming particles which exchanged energy and momentum and became two outgoing particles without any of the four worldlines intersecting. Thus it would appear that observer independence requires that we believe that physical interactions are non-local.

But if any two non-intersecting worldlines can exchange energy and momentum, generating two outgoing worldlines, which intersect neither with each other, nor with the incoming worldlines, then it would seem that any two particles in nature can interact with the same probabilities of those whose worldlines intersect. So physics becomes completely non-local. This is the catastrophe claimed in [6].

But we should be careful before jumping to this conclusion. What is true is that the concept of locality has to be altered to allow for the observer dependence of the notion that worldlines coincide in spacetime. But this need not imply that the notion of locality becomes so weakened that they theory can be ruled out by our store of observations that we usually take to support

the postulate that physical interactions are local. Might it be possible that the concept of locality is weakened or altered in a way that incorporates the observer dependence of worldline coincidence, but in a way that still restricts physical interactions in a way that is consistent with observations?

To investigate this we have to formulate exactly what is the concept of locality that is allowed by the structure of physics in κ -Minkowski spacetime. In this regard, I propose for consideration the following notion of locality, which I will call κ -locality. I define it as follows. $N \geq 3$ events E_I^a , $I = 1, 2, 3, \dots$ each on the worldline of a particle, $x_I^a(\tau)$ are κ -local if there is a passive Lorentz transformation to the description seen by another observer, which maps these events to a single event, p^a . ($x_I^a(\tau) = (t, x_I^i(\tau))$). Since each event is on a different worldline the Lorentz transformation leads them to intersect at a common point. Thus we say that the worldlines which do not intersect, but which can be mapped to intersecting worldlines by an energy and momentum dependent Lorentz transformation are κ -intersecting.

We then make a postulate which is a consequence of observer independence: *Suppose that there are $N \geq 3$ events, on N worldlines, which are κ -local. Then an interaction will take place with the same probability as it is seen to take place in the frame of the observer who sees the three events mapped to a single event, which is a coincidence of all three worldlines.* We can call this the hypothesis of κ -locality of physics. This can be easily generalized to more than three events. Note that, for the interaction probability to be non-zero, in addition to the worldlines coinciding the observation of energy and momentum must be satisfied. So the condition of the worldlines being κ -intersecting is a necessary but not a sufficient condition for an interaction to take place.

We next ask whether the hypothesis of κ -locality could be consistent with observation. There will certainly be an inconsistency with observation if κ -locality is generic. By this I mean the following. Consider an arbitrarily chosen N -tuple of world-lines of particles in spacetime. Each is a solution of the equations of motion and so is specified by a point in the phase space of a free relativistic particle. Generically they do not intersect, (we assume for the moment that space has $d = 3$ dimensions, below we consider the special case of $d = 1$.) Now we say that κ -locality is generic if for every such triple of particle worldlines there is an inertial observer who sees them to intersect at a point. That is, there are $N \geq 3$ events on the N worldlines that are taken to a single event under some momentum dependent Lorentz transformation.

If κ -locality is generic in this sense then the postulate of κ -locality of physics still does not imply that any $N \geq 3$ particles interact just as if they had an intersection. The conservation laws still have to be satisfied. But physics would still look very non-local because an interaction would be no more or less likely to happen than for the case of particles whose worldlines intersect at an event.

But suppose instead that κ -locality was a property of measure zero among triples of worldlines of particles. Then the situation is very different. In this case, we cannot use the fact that physical interactions have so far been found to be local to rule out the hypothesis that physics is instead κ -local. This is because to see that physics is κ -local rather than local we would have to very carefully prepare triples of particles on worldlines that have the property that a κ -local set can be made by selecting one event on each worldline. Since such experiments have never been made we cannot rule out the hypothesis that physics is κ -local. Indeed, it might be of interest to investigate how such experiments might be done.

Below I study separately the cases of $d = 1$ and $d = 3$ dimensions and show that in both cases κ -locality is a property of measure zero among triples or higher multiplets of worldlines. Hence I conclude that at least in κ -Minkowski spacetime the argument for a bound from observation fails.

At the end of this section, I consider the possibility that even if very rare, κ local interactions might contradict our understanding of very massive bodies like stars. The conclusion is that we cannot argue from existing experiments that we not live in κ -Minkowski spacetime.

5.1 Demonstration of non-genericity of κ -intersection for 3 or more particles in $d = 3$ spatial dimensions

I work here with the non-commuting coordinates (x^i, t) ¹⁰. In one observer's frame, called Alice, three particles described by classical worldlines coincide, an incoming photon, an incoming electron and an outgoing electron. From this local event we can certainly construct κ local triples of events. Under a Lorentz transformation to Bob's frame the event E is taken to three separated events, $E|_{\alpha}$, $\alpha = 1, 2, 3$ which are the image of E under the energy dependent Lorentz boosts for each of the particles.

So we can create κ -intersecting triples of worldlines by Lorentz transformations in this way from intersecting triples of worldlines. The question is, are such κ intersecting triples of worldlines, generic? That is given any three worldlines, $x_{\alpha}^a(\tau_{\alpha})$ in $3 + 1$ dimensional Minkowski spacetime, can we choose a triple of events, one on each of the world lines, $E_{\alpha}^a = x_{\alpha}^a(\tau_{\alpha})$ so that there is an energy dependent Lorentz transformation given by (37) and (38), below, that brings them to one event.

5.1.1 The set up

We can choose the gauge in which for each particle,

$$x_{\alpha}^0(\tau) = t_{\alpha}(\tau) = \tau_{\alpha} \quad (75)$$

Then the trajectories are given in terms of the initial conditions $(z_{\alpha}^i, p_{i\alpha})$ by

$$x_{\alpha}^i(t_{\alpha}) = z_{\alpha}^i + t_{\alpha} p_{i\alpha}, \quad (76)$$

After a Lorentz transformation to Bob's frame the trajectories are given by

$$x_{\alpha}^{it}(\tau_{\alpha}) = x_{\alpha}^i(\tau_{\alpha}) + \delta x_{\alpha}^i = z_{\alpha}^i + t_{\alpha} p_{i\alpha} + \delta x_{\alpha}^i \quad (77)$$

Similarly, the new time components are

$$t'_{\alpha}(\tau_{\alpha}) = t_{\alpha}(\tau_{\alpha}) + \delta t_{\alpha}(\tau_{\alpha}) \quad (78)$$

Here δx_{α}^i and δt_{α} are functions of the three boost parameters ω^i plus the initial conditions $(z_{\alpha}^i, p_{i\alpha})$, by (37) and (38)

These expressions (37,38) refer to Lorentz boosting around the origin of coordinates. But we can first do a translation, then boost. In κ -Poincare there is also a deformation of the action of translations, these are given by [12, 13],

$$\delta t = a^0 + t_p p_i a^i e^{-t_p p_0}; \quad \delta x^i = a^i \quad (79)$$

¹⁰It is easy to verify that the same conclusions apply to the case of intersecting worldlines in the commuting spacetime (x^i, T) .

for infinitesimal translations labeled by a four vector, (a^0, a^i) .

If we combine a translation of the origin with a Lorentz boost we have $\delta x^a = (\delta t, \delta x^i)$, given by

$$\delta x^i = -\omega^i(t + a^0 + t_p p \cdot a e^{-t_p p_0}) - t_p \epsilon^{ijk} \omega_j (L_k - \epsilon_{klm} a^l p^m) \quad (80)$$

and

$$\delta t = -\omega \cdot (\vec{x} + \vec{a})[1 + t_p E] + t_p p \cdot a e^{-t_p p_0} (t + a^0 + t_p p \cdot a e^{-t_p p_0}) O(t_p^2) \quad (81)$$

We note that rotations add nothing as they do not depend on energy or momentum.

5.1.2 Bringing three or more worldlines to intersect

We start with the case of three generic worldlines and ask whether there exists generically a Lorentz transformation that will bring them to intersect. Once we have seen that this can't generically be done we will see that as we increase the number of worldlines the constraints also increase.

We want to know if generically there exist three points, one on each wordline, $E_\alpha^a = x_\alpha^a(t_\alpha)$ such that there is a Lorentz transformation that brings them together to a single event. Lorentz transformations are parameterized generically by a boost parameter ω^i and a translation a^a . Given these there are three difference vectors

$$\Delta x_{\alpha\beta}^a = E_\alpha^a - E_\beta^a \quad (82)$$

of which any two, say Δx_{12}^a and Δx_{13}^a , are independent. Let us define the results of Lorentz transformations to be

$$\delta x_{\alpha\beta}^a = \delta x_\alpha^a - \delta x_\beta^a, \quad (83)$$

If the three points are brought together then

$$E_\alpha^a + \delta x_\alpha^a = E_\beta^a + \delta x_\beta^a \quad (84)$$

or

$$\Delta x_{\alpha\beta}^a = \delta x_{\alpha\beta}^a \quad (85)$$

5.1.3 A simplifying assumption

First we run the argument with a simplifying assumption. Below we will see the result is the same if we drop it, so this is just to make the reasons for the result clear.

Let us suppose that there is a Lorentz frame in which the three E_α^a are simultaneous. Let us first transform to that frame of reference. This means that the three t_α are equal to each other and hence all equal to a single time, s . Hence, the three $\Delta t_{\alpha\beta} = 0$.

It follows that

$$\delta x_{\alpha\beta}^i = -t_p \epsilon^{ijk} \omega_j (\Delta L_{k\alpha\beta} - \epsilon_{klm} a^l \Delta p_{\alpha\beta}^m) - t_p \omega^i a_k \Delta p_{\alpha\beta}^k \quad (86)$$

Here $\Delta L_{k\alpha\beta} = L_{k\alpha} - L_{k\beta}$ and similarly for $\Delta p_{\alpha\beta}^i$.

We also have

$$\delta t_{\alpha\beta} = -\omega \cdot (\vec{x}_\alpha[1 + t_p E_\alpha] - \vec{x}_\beta[1 + t_p E_\beta] + \vec{a} t_p [E_\alpha - E_\beta]) + t_p \omega_i (p_\alpha^i - p_\beta^i) s \quad (87)$$

Now we notice that

$$\omega_i \delta x_{\alpha\beta}^i = -t_p \omega \cdot \omega a_k \Delta p_{\alpha\beta}^k \quad (88)$$

Let us at first consider that the velocity of the boost is also small so we can neglect $\omega \cdot \omega$ and work only to leading order in $t_p |\omega|$. Then to this order, we can take from (85)

$$\omega_i \Delta x_{\alpha\beta}^i = O(t_p |\omega|^2) \quad (89)$$

This means that to leading order in $t_p |\omega|$, ω^i is normal to the plane formed by Δx_{12}^i and Δx_{13}^i . Thus, to this order, ω^i can have just a single component. If the unit vector proportional to that plane is \hat{n}^i then $\omega^i = \theta \hat{n}^i$ for some θ .

Let us now count the parameters to be varied and the equations to be solved. We are given the three worldlines, which fixes the trajectories (in the frame where all the E_α^0 's are equal) and the momenta and energies. For each time s these fix the $\Delta x_{\alpha\beta}^i$. We had originally to solve eight equations, six of the form

$$\delta x_{12}^i = \Delta x_{12}^i, \quad \delta x_{13}^i = \Delta x_{13}^i \quad (90)$$

and two more

$$\delta t_{12} = 0, \quad \delta t_{13} = 0, \quad (91)$$

However, our simplifying assumption has by (88) already solved one component each of the two equations in (90). Hence, to the order we are working, there are six more equations to solve.

The $\delta x_{\alpha\beta}^i$ and $\delta t_{\alpha\beta}$ are functions of one θ and three translation components, a^i . Since we are varying also s that gives us 5 degrees of freedom, to vary to look for a simultaneous solution to 6 equations. This is one more equations than variables to vary so if there are any solutions they are highly non-generic.

5.1.4 Dropping the simplifying assumption

Now we show we can drop the simplifying assumption and get the same answer. So we do not assume that the events are to begin with simultaneous and we also include terms of order $t_p |\omega|^2$.

We now have to solve in place of (90), a more complicated expression

$$\delta x_{\alpha\beta}^i = -\omega^i (t_\alpha - t_\beta + t_p a_k \Delta p_{\alpha\beta}^k) - t_p \epsilon^{ijk} \omega_j (\Delta L_{k\alpha\beta} - \epsilon_{klm} a^l \Delta p_{\alpha\beta}^m) = \Delta x_{\alpha\beta}^i \quad (92)$$

Then there will be components of ω^i besides $\theta \hat{n}^i$. Let us then write

$$\omega^i = \phi_{12} \Delta x_{12}^i + \phi_{13} \Delta x_{13}^i + \theta \hat{n}^i \quad (93)$$

From these we can compute the components $\Delta x_{12}^i \omega_i$ and $\Delta x_{13}^i \omega_i$. These are functions of the ϕ_{12}, ϕ_{13} and the norms and dot products of the Δx_{12}^i and Δx_{13}^i , which are already fixed. But they also have to satisfy

$$\omega_i \Delta x_{12}^i = -\omega^i \omega_i (t_1 - t_2 + t_p a_k (p_1^k - p_2^k)), \quad \omega_i \Delta x_{13}^i = -\omega^i \omega_i (t_1 - t_3 + t_p a_k (p_1^k - p_3^k)) \quad (94)$$

These two equations determine the two remaining components ϕ_{12} and ϕ_{13} in terms of the time differences $(t_1 - t_2)$ and $(t_1 - t_3)$. Once this is done the counting is as follows. We now have six

equations to satisfy, these are the 2 components of the 2 equations (90) in the plane orthogonal to ω , plus two more equations

$$\delta t_{12} = t_1 - t_2, \quad \delta t_{13} = t_1 - t_3, \quad (95)$$

We have five remaining degrees of freedom: θ , a^i and the one free time, t_1 to vary to solve these 6 equations. Five free variables is not enough to solve six equations, so again there remain no generic solutions.

It might help to make a remark about the counting. Initially we have 8 equations to solve (90,95). It seems at first that we have 9 free variables with which to solve them. These are three boost parameters, ω^i , three times t_α and three components of translations a^i (the time components of translations doesn't come into any of the equations to be solved.) However in equation (94) we see that two of the boost parameters are used up to compensate for the three times being unequal because ϕ_{12} and ϕ_{13} are solved in terms of $t_1 - t_2$ and $t_1 - t_3$. That is to say that we have two relations amongst these four variables, so only two of them are free. This uses up two of the eight equations, and leaves us with six equations to solve with only five remaining unknowns, θ , a^i and t_1 . This has generically no solution.

It is easy to see that if we increase the numbers of particles to four or more the counting gets rapidly worse. Suppose we add a fourth worldline, which we want to make intersect with the first three by the same Lorentz transformation. We are interested in the generic case, so we assume that in the original frame of reference this fourth worldline does not coincide with any of the first three. We want to solve the problem of whether there is a Lorentz transformation that can make all four worldlines coincide at some common event; clearly this only has a solution if that Lorentz transform already brings the first three worldlines to coincidence. So we can count the additional degrees of freedom and additional equations to be solved. There is one more degree of freedom, which is the time on the new worldline, t_4 . But there are four equations to be solved for each value of t_4 if the event $x_4^a(t_4)$ is to be brought by the Lorentz transformation to the intersection point the other three are brought to. So there are a total of three more constraints for a total of ten equations to be solved by six free variables. The situation is that κ -intersecting quadruples of worldlines are even less generic.

5.2 The case of $d = 1$ spatial dimension

The analysis in [6] is carried out in the $1 + 1$ dimensional case. We have to analyze that case separately because there is a special situation, which is that every two non-parallel worldlines meet somewhere. This means every triplet of worldlines has three generically three intersection points where they meet in pairs. Let us call the events where worldlines $x_\alpha^a(\tau)$ and $x_\beta^a(\tau)$ meet $z_{\alpha\beta}^a$.

The phase space is initially six dimensional.

Let us start with the case where three worldlines intersect at a single point,

$$p^a = z_{12}^a = z_{23}^a = z_{13}^a. \quad (96)$$

The condition that all three intersection points are equal is four equations. So the set of histories in which this is the case is a 2 dimensional subspace of the phase space. Indeed, it is parameterized by the points p^a of the intersection, which is two dimensional.

Now the set of Lorentz boosts is in this case 3 dimensional, one for the single boost parameter, and two for the origin around which the boost is made. However of the two degrees of freedom that come from translating the boosts, we have already seen from (38) that only one comes into the energy dependence of the Lorentz transformations. So we have four equations to solve with two parameters. (The freedom to choose the τ_α is taken into account here by the fact that there are generic intersection points of each pair we are trying to bring together at one event.)

So the question is, can the four equations that represent the coincidence of the three intersection points be generated from a generic point by the two degrees of freedom of Lorentz boosts. The answer is generically no. That is, consider a generic history with three worldlines that meet in pairs. This is a six dimensional set. They cannot be all brought to an element of the two dimensional set where all three coincide at a single point by a three parameter family of Lorentz boosts.

So we see that even in $d = 1$ dimensions having three κ -intersecting worldlines are a non-generic set of measure zero.

There is another way to see this. If we look at (37) for $d = 1$ we see that the term in L^i drops out and there is just the ordinary Lorentz transformation, which is not energy dependent. The only energy dependence in the Lorentz transformations is the term (81) we have already talked about. So consider three generic worldlines, which intersect in pairs to give three intersection points. The only thing that the energy dependence of the Lorentz transformations can do is to move the three worldlines up or down by (81). The three worldlines move by different amounts due to the different E_α 's, but all three are determined by one parameter which is $\omega \cdot a$. The condition that the three intersection points shrink to one is four equations, but one only has a single parameter to tune to satisfy them. Generically there will be no choice of $\omega \cdot a$ that reduces the three intersection points to a single triple intersection point.

5.3 Stars and so forth

One might worry that the non-localities inherent in *DSR* theories will change the bulk properties of large collections of matter such as stars in a way that causes our observations of them to disagree with theory¹¹. That is, even if κ -intersecting triples of worldlines are rare, there are so many triples of atoms in stars that some of them will be sometimes κ -local, so some κ -local interactions will take place.

Let us grant this for the moment. The problem is, how this is to be observed? A star is already in thermal equilibrium from a myriad of local interactions. These have mainly the effect of reproducing the thermal distribution of the atoms in the star, which means they are completely randomized. If we throw in a small proportion of non-local interactions this is not going to have a measurable effect on the state of thermal equilibrium. Since the κ -local interactions are, by the postulate of observer independence, of the same kinds as local interactions, nothing is going to happen that doesn't already happen much more numerous with local interactions. They are not going to change the temperature, pressure or equation of state of the interior of the star by a measurable amount.

There are in fact models of statistical mechanical systems on lattices that are modified by the addition of a small number of non-local interactions[24]. There are observable effects when the

¹¹Sabine Hossenfelder, personal communication.

proportion of non-local effects are high enough, but this is very unlikely to be the case here. Moreover the effects that do occur are subtle and not easily measured, one for example is a slight change in the temperature of phase transitions. It is hard to imagine making a measurement that could detect non-local interactions through a small change in the temperature of some phase transition in the nuclear matter making up the star, as this is already likely to be in phenomena already parameterized by some phenomenological model of the equation of state of the gas or nuclear matter making up the star or neutron star.

In any case, to study the possibility of observable effects due to κ -locality is a research problem, which would be interesting to carry out. Given that the result is very unlikely to be significant, because that κ intersecting triples are a set of measure zero, it would be premature to speculate that any observable effect is produced.

6 Conclusions

The main aim of this paper has been to show how paradoxes of locality can be generated in at least one approach to *DSR*-that based on free particles in κ -Minkowski spacetime-but only at the classical level, and to propose that when the quantum dynamics are taken into account those paradoxes may be resolved. I was able to provide partial evidence, but not proof, in support of this hypothesis.

I also investigated the possibility that the notion of locality be generalized to a notion appropriate to the physics of classical particles described by worldlines in the non-commutative geometry of κ -Minkowski spacetime.

The conclusions can, in more detail, be stated as follows.

- In section 3.4 we showed how to generate classical paradoxes of locality, by using the dependence of the Lorentz transformations on energy, momentum and angular momentum. We considered a case of two or more particles which coincide in one observers frame, and found expressions for their differences in space and time coordinates as seen in a second frame, given by equations (45) and (46).
- In section 4.4.1 we set up the quantum theory for a free relativistic particle in κ -Minkowski spacetime, computed the propagation of a Gaussian wavepacket, defined initially in momentum space, transformed to spacetime, and found that there is an anomalous spreading of the wavepacket given by (62). This is proportional to $l_p x \Delta p / \hbar$.

Thus, we can conclude that the classical paradox is resolved for quantum states in which $\Delta p \geq |p|$. On the other hand, if $\Delta p \leq |p|$ then the paradoxes may remain. What is the meaning of this?

Since we are working with a gaussian wavepacket, and we can assume that $E \ll E_p$, we find that the paradoxes are resolved so long as $\Delta x < \lambda$, where $\lambda = \hbar/p$ is the wavelength of light. This means that the particle picture is preferred to the wave picture because the photon (or massless particle) can be localized to within its wavelength. We may argue that this is the regime where a single propagating photon can be treated as a quantum free relativistic particle, which is the approximation we are employing here.

We can speculate that the other case, where $\Delta x > \lambda$ requires the full treatment of the quantum field theory to resolve. We remark that there are known to be additional sources of non-locality coming from interactions in non-commutative field theory, which may play a role in the resolution of the paradoxes in the full interacting quantum field theory. We may hypothesize that a notion like κ -locality may be relevant there.

- I proposed in section 5 a weakening of locality to κ -locality defined by worldlines intersecting in the frame of some inertial observer. I showed that this property is very non-generic, so it is the case that only a set of measure zero of $N \geq 3$ worldlines κ -interact.

So, to conclude, we have shown that there is a regime where quantum effects suffice to resolve the paradoxes, but we have not so far shown this in general.

We note that the problem of measurements and uncertainty in κ -Minkowski spacetime has been studied before[17], and the effect of spreading of wavepackets has been noted[18, 19]. However these previous papers do not seem to have studied propagation in the (x^i, T) commuting coordinates. The novel assertion of the present work then appears to be the insistence that the spacetime we experience must be a commutative spacetime, which means that quantum state evolve in space according to the time coordinate T which is different from the time coordinate t in which we can study evolution in momentum space.

There are in addition a number of questions and issues that remain to be sorted out in this area.

- The commutation relations (4) and (10) are not invariant under time reversal ($t \rightarrow -t$ or $T \rightarrow -T$, respectively). Another way to say this is that the theory seems physically different depending on whether the sign of t_p is taken negative, or positive as we have taken it here. Are there physical consequences of this?
- Similarly, the theory does not seem invariant under the exchange of E for $-E$. Is there still *CPT* symmetry?
- It appears that the phase velocity computed above is not equal to the classical computation of dx^i/dT computed via the Hamiltonian dynamics. Is this a problem?
- It is known that *QFT* defined on noncommutative geometries in which time and space coordinates don't commute are not unitary[20]. Might it be that unitary evolution could be defined instead in the time variable T ?
- There is a long standing debate as to whether the physics in κ -Minkowski spacetime should be invariant, not just under Lorentz transformations, but under non-linear redefinitions of the phase space coordinates. A view that seems reasonable is that *DSR* theories will be limits of quantum theories of gravity, whose physics will define the appropriate notion of locality and hence the physical spacetime. Thus, a particular set of choices for physical momenta, energy and spacetime coordinates will be reflected in the form of the Hamiltonian. At the same time, any quantum mechanical system is invariant under changes of basis in Hilbert space. It may be that the observer independence provided by choices of basis in Hilbert space implies an expansion of the notion of observer independence in a quantum theory of gravity, but this is a subtle issue that goes beyond the scope of this paper.

Finally, it is important to remark that even if the proposed apparent paradoxes are in the end resolved, the result is extremely important because the quantum effects necessary to resolve them introduce a stochastic influence of energy on arrival times. This is relevant for experiments that test the possible energy dependence of the speed of light. For this reason the paradoxes proposed in [5, 6] are playing a central role in the development of our understanding of the possible observational consequences of quantum gravity.

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