Convergent Multiplicative Processes Repelled from Zero: Power Laws and Truncated Power Laws

Didier Sornette ([1,2,*]) and Rama Cont ([1])

([1]) Laboratoire de Physique de la Matière Condensée ([**]), Université des Sciences, BP 70, Parc Valrose, 06108 Nice Cedex 2, France
([2]) Department of Earth and Space Science, and Institute of Geophysics and Planetary Physics, University of California, Los Angeles, California 90095, USA

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Abstract. — Levy and Solomon have found that random multiplicative processes $w_t = \lambda_1 \lambda_2 ... \lambda_t$ (with $\lambda_j > 0$) lead, in the presence of a boundary constraint, to a distribution $P(w_t)$ in the form of a power law $w_t^{-(1+\mu)}$. We provide a simple exact physically intuitive derivation of this result based on a random walk analogy and show the following: 1) the result applies to the asymptotic ($t \to \infty$) distribution of w_t and should be distinguished from the central limit theorem which is a statement on the asymptotic distribution of the reduced variable $\frac{1}{\sqrt{t}}(\log w_t - \langle \log w_t \rangle)$; 2) the two necessary and sufficient conditions for $P(w_t)$ to be a power law are that $\langle \log \lambda_j \rangle < 0$ (corresponding to a drift $w_t \to 0$) and that w_t not be allowed to become too small. We discuss several models, previously thought unrelated, showing the common underlying mechanism for the generation of power laws by multiplicative processes: the variable $\log w_t$ undergoes a random walk repelled from $-\infty$, which we describe by a Fokker-Planck equation. 3) For all these models, we obtain the exact result that μ is solution of $\langle \lambda^\mu \rangle = 1$ and thus depends on the distribution of λ. 4) For finite t, the power law is cut-off by a log-normal tail, reflecting the fact that the random walk has not the time to scatter off the repulsive force to diffusively transport the information far in the tail.

Résumé. — Levy et Solomon ont montré qu'un processus multiplicatif du type $w_t = \lambda_1 \lambda_2 ... \lambda_t$ (avec $\lambda_j > 0$) conduit, en présence d'une contrainte de bord, à une distribution $P(w_t)$ en loi de puissance $w_t^{-(1+\mu)}$. Nous proposons une dérivation simple, intuitive et exacte de ce résultat basée sur une analogie avec une marche aléatoire. Nous obtenons les résultats suivants: 1) le régime de loi de puissance décrit la distribution asymptotique de w_t aux grands temps et doit être distingué du théorème limite central décrivant la convergence de la variable réduite $\frac{1}{\sqrt{t}}(\log w_t - \langle \log w_t \rangle)$ vers la loi Gaussienne; 2) les deux conditions nécessaires et suffisantes pour que $P(w_t)$ soit une loi de puissance sont $\langle \log \lambda_j \rangle < 0$ (correspondant à une dérive vers zéro) et la contrainte que w_t soit empêchée de trop s'approcher de zéro. Cette contrainte peut être mise en oeuvre de manière variée, généralisant à une grande classe de modèles le cas d'une barrière réfléchissante examiné par Levy et Solomon. Nous donnons aussi un traitement approximatif, devenant exact dans

(*) Author for correspondence (e-mail: sornette@naxos.unice.fr)
(**) CNRS URA 190

la limite où la distribution de λ est étroite ou log-normale en terme d'équation de Fokker-Planck. 3) Pour tous ces modèles, nous obtenons le résultat général exact que l'exposant μ est la solution de l'équation $\langle \lambda^\mu \rangle = 1$. μ est donc non-universel et dépend de la spécificité de la distribution de λ. 4) Pour des t finis, la loi de puissance est tronquée par une queue log-normale due à une exploration finie de la marche aléatoire.

1. Introduction

Many mechanisms can lead to power law distributions. Power laws have a special status due to the absence of a characteristic scale and the implicit (to the physicist) relationship with critical phenomena, a subtle many-body problem in which self-similarity and power laws emerge from cooperative effects leading to non-analytic behavior of the partition or characteristic function.

Recently, Levy and Solomon [1] have presented a novel mechanism based on random multiplicative processes:

$$w_{t+1} = \lambda_t w_t, \tag{1}$$

where λ_t is a stochastic variable with probability distribution $\Pi(\lambda_t)$ and we express w_t in units of a reference value $w_{\rm u}$ which could be of the form e^{rt}, with r constant. All our analysis below then describes the distribution of w_t normalized to $w_{\rm u}$, in other words in the "reference frame" moving with $w_{\rm u}$. At the end, we can easily make reappear the scale $w_{\rm u}$ by replacing everywhere w by $w/w_{\rm u}$.

Taken literally with no other ingredient, expression (1) leads to the log-normal distribution [2–4]. Indeed, taking the logarithm of (1), we can express the distribution of $\log w$ as the convolution of t distributions of $\log \lambda$. Using the cumulant expansion and going back to the variable w_t leads, for large times t, to

$$P(w_t) = \frac{1}{\sqrt{2\pi Dt}} \frac{1}{w_t} \exp\left[-\frac{1}{2Dt}(\log w_t - vt)^2\right], \tag{2}$$

where $v = \langle \log \lambda \rangle \equiv \int_0^\infty {\rm d}\lambda \, \log \lambda \Pi(\lambda)$ and $D = \langle (\log \lambda)^2 \rangle - \langle \log \lambda \rangle^2$. Expression (2) can be rewritten

$$P(w_t) = \frac{1}{\sqrt{2\pi Dt}} \frac{1}{w_t^{1+\mu(w_t)}} e^{\mu(w_t)vt} \tag{3}$$

with

$$\mu(w_t) = \frac{1}{2Dt} \log \frac{w_t}{e^{vt}}. \tag{4}$$

Since $\mu(w_t)$ is a slowly varying function of w_t, this form shows that the log-normal distribution can be mistaken for an apparent power law with an exponent μ slowly varying with the range w_t which is measured. Indeed, it was pointed out [5] that for $w_t \ll e^{(v+2D)t}$, $\mu(w_t) \ll 1$ and the log-normal is undistinguishable from the $1/w_t$ distribution, providing a mechanism for $1/f$ noise. However, notice that $\mu(w_t) \to \infty$ far in the tail $w_t \gg e^{(v+2D)t}$ and the log-normal distribution is *not* a power law.

The ingredient added by Levy and Solomon [1] is to constrain w_t to remain larger than a minimum value $w_0 > 0$. This corresponds to put back w_t to w_0 as soon as it would become smaller. To understand intuitively what happens, it is simpler to think in terms of the variables $x_t = \log w_t$ and $l = \log \lambda$, here following [1]. Then obviously, the equation (1) defines a random walk in x-space with steps l (positive and negative) distributed according to the density

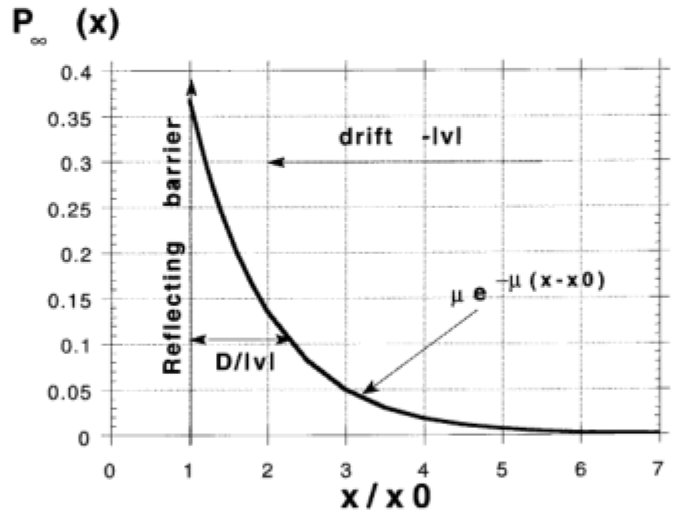

Fig. 1. — Steady-state exponential profile of the probability density of presence of the random walk with a negative drift and a reflecting barrier.

distribution $\pi(l) = \mathrm{e}^l \Pi(\mathrm{e}^l)$. The distribution of the position of the random walk is similarly defined: $\mathcal{P}(x_t, t) = \mathrm{e}^{x_t} P(\mathrm{e}^{x_t}, t)$.

• If $v \equiv \langle l \rangle = \langle \log \lambda \rangle > 0$, the random walk is biased and drifts to $+\infty$. As a consequence, the presence of the barrier has no important consequence and we recover the log-normal distribution (2) apart from minor and less and less important boundary effects at $x_0 = \log w_0$, as t increases. Thus, this regime is without surprise and does not lead to any power law. We can however transform this case in the following one $v \equiv \langle l \rangle < 0$ with a suitable definition of the moving reference scale $w_{\mathrm{u}} \sim \mathrm{e}^{rt}$ such that, in this frame, the random random drifts to the left. But the barrier has to stay fixed in the moving frame, corresponding to a moving barrier in the unscaled variable w_t.

• If $v \equiv \langle l \rangle < 0$, the random walk drifts towards the barrier. The qualitative picture is the following (see Figs. 1 and 2): a steady-state ($t \to \infty$) establishes itself in which the net drift to the left is balanced by the reflection on the reflecting barrier. The random walk becomes trapped in an effective cavity of size of order D/v with an exponential tail (see below). Its incessant motion back and forth and repeated reflections off the barrier and diffusion away from it lead to the build-up of an exponential probability (concentration) profile (and no more a Gaussian). The probability density function of the walker position x is then of the form $\mathrm{e}^{-\mu x}$ with $\mu \approx |v|/D$. As x is the logarithm of the random variable w, then one obtains a power law distribution for w of the form $\sim w^{-(1+\mu)}$.

We first present an intuitive approximate derivation of the power law distribution and its exponent, using the Fokker-Planck formulation in a random walk analogy. In Section 2.2, the problem is formulated rigorously and solved exactly in Section 2.5. Sections 2.3 and 2.4 are generalization of the process (1). The explicit calculation of the exponent of the power law distribution is done using a Wiener-Hopf integral equation, showing that it is controlled by extreme values of the process.

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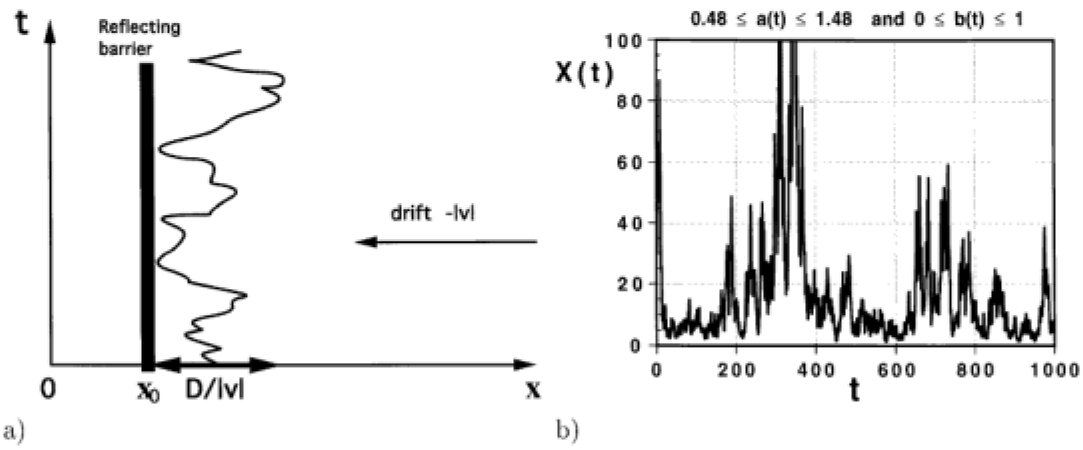

Fig. 2. — a) A typical trajectory of the random walker at large times showing the multiple reflections off the barrier. b) The time evolution of the Kesten variable defined by the equation (19) with a_t uniformly taken in the interval $[0.48; 1.48]$ leading to $\mu \approx 1.47$ according to (17) and b_t uniformly taken in the interval $[0; 1]$. Notice the intermittent large excursions.

2. The Random Walk Analogy

In the $x_t = \log w_t$ and $l_t = \log \lambda_t$ variables, expression (1) reads

$$x_{t+1} = x_t + l_t, \tag{5}$$

and describes a random walk with a drift $\langle l \rangle < 0$ to the left. The barrier at $x_0 = \log w_0$ ensures that the random walk does not escape to $-\infty$. This process is described by the Master equation [1]

$$\mathcal{P}(x, t+1) = \int_{-\infty}^{+\infty} \pi(l) \mathcal{P}(x-l, t) \mathrm{d}l. \tag{6}$$

2.1. Perturbative Analysis. — To get a physical intuition of the underlying mechanism, we now approximate this exact Master equation by its corresponding Fokker-Planck equation. Usually, the Fokker-Planck equation becomes exact in the limit where the variance of $\pi(l)$ and the time interval between two steps go to zero while keeping a constant finite ratio defining the diffusion coefficient [6]. In our case, this corresponds to taking the limit of very narrow $\pi(l)$ distributions. In this case, we can expand $\mathcal{P}(x-l, t)$ up to second order

$$\mathcal{P}(x-l, t) = \mathcal{P}(x, t) - l \frac{\partial \mathcal{P}}{\partial x}|_{(x,t)} + \frac{1}{2} l^2 \frac{\partial^2 \mathcal{P}}{\partial x^2}|_{(x,t)}$$

leading to the Fokker-Planck formulation

$$\frac{\partial \mathcal{P}(x,t)}{\partial t} = -\frac{\partial j(x,t)}{\partial x} = -v \frac{\partial \mathcal{P}(x,t)}{\partial x} + D \frac{\partial^2 \mathcal{P}(x,t)}{\partial x^2}, \tag{7}$$

where $v = \langle l \rangle$ and $D = \langle l^2 \rangle - \langle l \rangle^2$ are the leading cumulants of $\Pi(\log \lambda)$. $j(x,t)$ is the flux defined by

$$j(x,t) = v \mathcal{P}(x,t) - D \frac{\partial \mathcal{P}(x,t)}{\partial x}. \tag{8}$$

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Expression (7) is nothing but the conservation of probability. It can be shown that this description (7) is generic in the limit of very narrow π distributions: the details of π are not important for the large t behavior; only its first two cumulants control the results [6]. v and D introduce a characteristic "length" $\hat{x} = D/|v|$. In the overdamped approximation, we can neglect the inertia of the random walker, and the general Langevin equation $m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\gamma\frac{\mathrm{d}x}{\mathrm{d}t} + F + F_{\mathrm{fluct}}$ reduces to

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v + \eta(t), \tag{9}$$

which is equivalent to the Fokker-Planck equation (7). η is a noise of zero mean and delta correlation with variance D. This form exemplifies the competition between drift $v = -|v|$ and diffusion $\eta(t)$.

The stationary solution of (7), $\frac{\partial \mathcal{P}(x,t)}{\partial t} = 0$, is immediately found to be

$$\mathcal{P}_\infty(x) = A - \frac{B}{\mu}\mathrm{e}^{-\mu x}, \tag{10}$$

with

$$\mu \equiv \frac{|v|}{D}. \tag{11}$$

A and B are two constants of integration. Notice that, as expected in this approximation scheme, μ is the inverse of the characteristic length $\hat{x}$. In absence of the barrier, the solution is obviously $A = B = 0$ leading to the trivial solution $P_\infty(x) = 0$, which is indeed the limit of the log-normal form (2) when $t \to \infty$. In the presence of the barrier, there are two equivalent ways to deal with it. The most obvious one is to impose normalization

$$\int_{x_0}^{\infty} \mathcal{P}_\infty(x)\mathrm{d}x = 1, \tag{12}$$

where $x_0 \equiv \log w_0$. This leads to

$$\mathcal{P}_\infty(x) = \mu\mathrm{e}^{-\mu(x-x_0)}. \tag{13}$$

Alternatively, we can express the condition that the barrier at x_0 is *reflective*, namely that the flux $j(x_0) = 0$. Let us stress that the correct boundary condition is indeed of this type (and not absorbing for instance) as the rule of the multiplicative process is that we put back w_t to w_0 when it becomes smaller than w_0, thus ensuring $w_t \geq w_0$. An absorbing boundary condition would correspond to kill the process when $w_t \leq w_0$. Substituting (10) in (8) with $j(x_0) = 0$, we retrieve (13) which is automatically normalized. Reciprocally, (13) obtained from (12) satisfies the condition $j(x_0) = 0$.

There is a faster way to get this result (13) using an analogy with a Brownian motion in equilibrium with a thermal bath. The bias $\langle l \rangle < 0$ corresponds to the existence of a constant force $-|v|$ in the $-x$ direction. This force derives from the linearly increasing potential $V = |v|x$. In thermodynamic equilibrium, a Brownian particle is found at the position x with probability given by the Boltzmann factor $\mathrm{e}^{-\beta|v|x}$. This is exactly (13) with $D = 1/\beta$ as it should from the definition of the random noise modelling the thermal fluctuations.

Translating in the initial variable $w_t = \mathrm{e}^x$, we get the Paretian distribution

$$P_\infty(w_t) = \frac{\mu w_0^\mu}{w_t^{1+\mu}}, \tag{14}$$

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with μ given by (11):

$$\mu \equiv \frac{|\langle \log \lambda \rangle|}{\langle (\log \lambda)^2 \rangle - \langle \log \lambda \rangle^2}. \tag{15}$$

These two derivations should not give the impression that we have found the exact solution. As we show below, it turns out that the exponential form is correct but the value of μ given by (15) is only an approximation. As already stressed, the Fokker-Planck is valid in the limit of narrow distributions of step lengths. The Boltzmann analogy assumes thermal equilibrium, *i.e.* that the noise is distributed according to a Gaussian distribution, corresponding to a log-normal distribution for the λ's. These restrictive hypothesis are not obeyed in general for arbitrary $\Pi(\lambda)$. The power law distribution (14) is sensitive to large deviations not captured within the Fokker-Planck approximation.

2.2. Exact Analysis. — In the general case where these approximations do not hold, we have to address the general problem defined by the equations (5, 6). Let us consider first the case where the barrier is absent. As already stated, the random walk eventually escapes to $-\infty$ with probability one. However, it will wander around its initial starting point, exploring maybe to the right and left sides for a while before escaping to $-\infty$. For a given realization, we can thus measure the rightmost position $x_{\max}$ it ever reached over all times. What is the distribution $\mathcal{P}_{\max}(\mathrm{Max}(0, x_{\max}))$? The question has been answered in the mathematical literature using renewal theory ([7], p. 402) and the answer is

$$\mathcal{P}_{\max}((\mathrm{Max}(0, x_{\max})) \sim \mathrm{e}^{-\mu x_{\max}}, \tag{16}$$

with μ given by

$$\int_{-\infty}^{+\infty} \pi(l) \mathrm{e}^{\mu l} \mathrm{d}l = \int_{0}^{+\infty} \Pi(\lambda) \lambda^{\mu} \mathrm{d}\lambda = 1. \tag{17}$$

The proof can be sketched in a few lines [7] and we summarize it because it will be useful in the sequel. Consider the probability distribution function $M(x) \equiv \int_{-\infty}^{x} \mathcal{P}_{\max}(x_{\max}) \mathrm{d}x_{\max}$, that $x_{\max} \leq x$. Starting at the origin, this event $x_{\max} \leq x$ occurs if the first step of the random walk verifies $x_1 = y \leq x$ together with the condition that the rightmost position of the random walk starting from $-x_1$ is less or equal to $x - y$. Summing over all possible y, we get the Wiener-Hopf integral equation

$$M(x) = \int_{-\infty}^{x} M(x-y) \pi(y) \mathrm{d}y. \tag{18}$$

It is straightforward to check that $M(x) \to \mathrm{e}^{-\mu x}$ for large x with μ given by (17). We refer to [7] for the questions of uniqueness and to [9, 10] for classical methods for handling Wiener-Hopf integral equations. We shall encounter the same type of Wiener-Hopf integral equation in Section 2.5 below which addresses the general case.

How is this result useful for our problem? Intuitively, the presence of the barrier, which prevents the escape of the random walk, amounts to reinjecting the random walker and enabling it to sample again and again the large positive deviations described by the distribution (16). Indeed, for such a large deviation, the presence of the barrier is not felt and the presence of the drift ensures the validity of (16) for large x. These intuitive arguments are shown to be exact in Section 2.5 for a broad class of processes.

Let us briefly mention that there is another way to use this problem, on the rightmost position $x_{\max}$ ever reached, to get an exponential distribution and therefore a power law distribution in the w_t variable. Suppose that we have a constant input of random walkers, say at the origin. They establish a uniform flux directed towards $-\infty$. The density (number per unit length) of these walkers to the right is obviously decaying as given by (16) with (17). This provides an alternative mechanism for generating power laws, based on the superposition of many convergent multiplicative processes.

Let us now compare the two results (15, 17) for μ. It is straightforward to check that (15) is the solution of (17) when $\pi(l)$ is a Gaussian *i.e.* $\Pi(\lambda)$ is a log-normal distribution. (15) can also be obtained perturbatively from (17): expanding $e^{\mu l}$ as $e^{\mu l} = 1 + \mu l + \frac{1}{2}\mu^2 l^2 + ...$ up to second order and re-exponentiating, we find that the solution of (17) is (15). This was expected from our previous discussion of the approximation involved in the use of the Fokker-Planck equation.

2.3. Relation with Kesten Variables. — Consider the following mixture of multiplicative and additive process defining a random affine map:

$$S_{t+1} = b_t + \lambda_t S_t, \tag{19}$$

with λ and b being positive independent random variables. The stochastic dynamical process (19) has been introduced in various occasions, for instance in the physical modelling of 1D disordered systems [11] and the statistical representation of financial time series [12]. The variable $S(t)$ is known in probability theory as a Kesten variable [13].

Consider as an example the number of fish S_t in a lake in the t-th year. The population S_{t+1} in the $(t+1)$st year is related to the population S_t through (19). The growth rate λ_t depends on the rate of reproduction and the depletion rate due to fishing as well as environmental conditions, and is therefore a variable quantity. The quantity b_t describes the input due to restocking from an external source such as a fish hatchery in artificial cases, or from migration from adjoining reservoirs in natural cases. This model (19) can be applied to the problems of population dynamics, epidemics, investment portfolio growth, and immigration across national borders [8]. The justification of our interest in (19) relies on the fact that it is the simplest *linear* stochastic equation that can provide an alternative modelling strategy for describing complex time series to the nonlinear deterministic maps. Notice that the multiplicative process, with a λ_t that can take values larger than 1, ensures an intermittent sensitive dependence on initial conditions. The restocking term b_t, or more generally the repulsion from the origin, corresponds to a reinjection of the dynamics. It is noteworthy that these two ingredients, of sensitive dependence on initial conditions and reinjection, are also the two fundamental properties of systems exhibiting chaotic behavior.

$b = 0$ recovers (1) (without the barrier). For $b \neq 0$, it is well-known that for $\langle \log \lambda \rangle < 0$, $S(t)$ is distributed according to a power law

$$P(S_t) \sim S_t^{-(1+\mu)}, \tag{20}$$

with μ determined by the condition (17) [13] already encountered above $\langle \lambda^\mu \rangle = 1$. In fact, the derivation of (20) with (17) uses the result (16) of the renewal theory of large positive excursions of a random walk biased towards $-\infty$ [12]. Figure 3 shows the reconstructed probability density of the Kesten variable S_t for λ_t and b_t uniformly sampled in the interval $[0.48; 1.48]$ and in $[0, 1]$ respectively. This corresponds to the theoretical value $\mu \approx 1.47$. We have also constructed the probability density function of the variations $S_{t+1} - S_t$ of the Kesten variable for the same values. We observe again a power law tail for the positive and negative variations, with the same exponent.

This is not by chance and we now show that the multiplicative process with the reflective barrier and the Kesten variable are deeply related. First, notice that for $\langle \log \lambda \rangle < 0$ in absence

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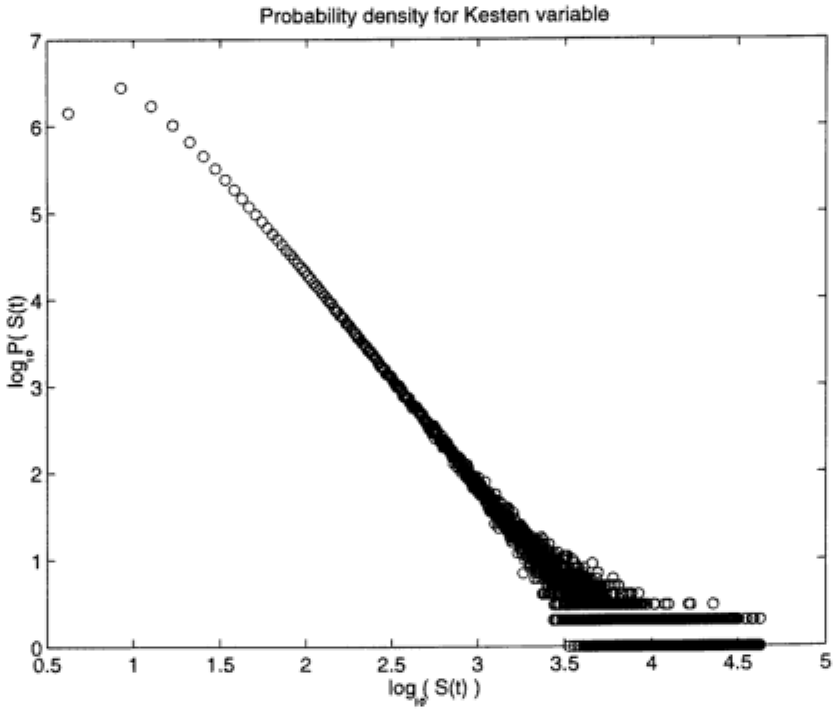

Fig. 3. — Reconstructed natural logarithm of the probability density of the Kesten variable S_t as a function of the logarithm of S_t, for $0.48 \leq \lambda_t \leq 1.48$ and $0 \leq b_t \leq 1$, uniformly sampled. The theoretical prediction $\mu \approx 1.47$ from (17) is quantitatively verified.

of $b(t)$, S_t would shrink to zero. The term $b(t)$ can be thought of as an effective repulsion from zero and thus acts similarly to the previous barrier w_0. To see this more quantitatively, we form

$$\frac{S_{t+1} - S_t}{S_t} = \frac{b_t}{S_t} + \lambda_t - 1. \tag{21}$$

We make the approximation of writing the finite difference $\frac{S_{t+1}-S_t}{S_t}$ as $\frac{\mathrm{d}\log S}{\mathrm{d}t}$. It has the same status as the one used to derive the Fokker-Planck equation and will lead to results correct up to the second cumulant. Introducing again the variable $x \equiv \log S$, expression (21) gives the overdamped Langevin equation:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = b(t)\mathrm{e}^{-x} - |v| + \eta(t), \tag{22}$$

where we have written $\lambda(t) - 1$ as the sum of its mean and a purely fluctuating part. We thus get $v = \langle\lambda\rangle - 1 \simeq \langle\log\lambda\rangle$ and $D \equiv \langle\eta^2\rangle = \langle\lambda^2\rangle - \langle\lambda\rangle^2 \simeq \langle\log(\lambda)^2\rangle - \langle\log\lambda\rangle^2$. Compared to (9), we see the additional term $b(t)\mathrm{e}^{-x}$, corresponding to a repulsion from the $x < 0$ region. This repulsion replaces the reflective barrier, which can itself in turn be modelled by a concentrated force. The corresponding Fokker-Planck equation is

$$\frac{\partial P(x,t)}{\partial t} = b(t)\mathrm{e}^{-x}P(x,t) - (v + b(t)\mathrm{e}^{-x})\frac{\partial P(x,t)}{\partial x} + D\frac{\partial^2 P(x,t)}{\partial x^2}. \tag{23}$$

It also presents a well-defined stationary solution that we can easily obtain in the regions $x \to +\infty$ and $x \to -\infty$. In the first case, the terms $b(t)e^{-x}$ can be neglected and we recover the previous results (13) with x_0 now determined from asymptotic matching with the solution at $x \to -\infty$. For $x \to -\infty$, we can drop all the terms except those in factor of the exponentials which diverge and get $P(x) \to e^x$. Back in the w_t variable, $P_\infty(S_t)$ is a constant for $S_t \to 0$ and decays algebraically as given by (14) with the exponent (11, 15) for $S_t \to +\infty$. Beyond these approximations, we can solve exactly expression (21) or equivalently (19) and we recover (17). This is presented in Section 2.5 below. Again, notice that (11, 15) is equal to the solution of (17) up to second order in the cumulant expansion of the distribution of $\log \lambda$.

It is interesting to note that the Kesten process (19) is a generalization of branching processes [14]. Consider the simplest example of a branching process in which a branch can either die with probability p_0 or give two branches with probability $p_2 = 1 - p_0$. Suppose in addition that, at each time step, a new branch nucleates. Then, the number of branches S_{t+1} at generation $t+1$ is given by equation (19) with $b_t = 1$ and $\lambda_t = \frac{2j_{t+1}}{S_t}$, where j_{t+1} is the number of branches out of the S_t which give two branches. The distribution $\Pi(\lambda)$ is simply deduced from the binomial distribution of j_{t+1}, namely $\binom{S_t}{j_{t+1}} p_2^{j_{t+1}} p_0^{S_t - j_{t+1}} \equiv \frac{[S_t]!}{[S_t - j_{t+1}]![j_{t+1}]!} p_2^{j_{t+1}} p_0^{S_t - j_{t+1}}$. For large S_t, $\Pi(\lambda)$ is approximately a Gaussian with a standard deviation equal to $\frac{4p_0(1-p_0)}{S_t}$, *i.e.* it goes to zero for large S_t. We thus pinpoint here the key difference between standard branching processes and the Kesten model: in branching models, large generations are *self-averaging* in the sense that the number of children at a given generation fluctuates less and less as the size of the generation increases, in contrast to equation (19) exhibiting the same *relative* fluctuation amplitude. This is the fundamental reason for the robustness of the existence of a power law distribution in contrast to branching models in which a power law is found only for the special critical case $p_0 = p_2$ (for $p_0 > p_2$, the population dies off, while for $p_0 < p_2$ the population prolifates exponentially). The same conclusion carries out directly for more general branching models. Note finally that it can be shown that the branching model previously defined becomes equivalent to a Kesten process if the number of branches formed from a single one is itself a random variable distributed according to a power law with the special exponent $\mu = 1$, ensuring the scaling of the fluctuations with the size of the generations.

2.4. GENERALIZATION TO A BROAD CLASS OF MULTIPLICATIVE PROCESS WITH REPULSION AT THE ORIGIN. — The above considerations lead us to propose the following generalization

$$w_{t+1} = e^{f(w_t, \{\lambda_t, b_t, ...\})} \lambda_t w_t, \tag{24}$$

where $f(w_t, \{\lambda_t, b_t, ...\})) \to 0$ for $w_t \to \infty$ and $f(w_t, \{\lambda_t, b_t, ...\})) \to \infty$ for $w_t \to 0$.

The model (1) is the special case $f(w_t, \{\lambda_t, b_t, ...\}) = 0$ for $w_t > w_0$ and $f(w_t, \{\lambda_t, b_t, ...\}) = \log(\frac{w_0}{\lambda_t w_t})$ for $w_t \le w_0$. The Kesten model (19) is the special case $f(w_t, \{\lambda_t, b_t, ...\}) = \log(1 + \frac{b(t)}{\lambda_t w_t})$. More generally, we can consider a process in which at each time step t, after the variable λ_t is generated, the new value $\lambda_t w_t$ (or $\lambda_t w_t + b_t$ in the case of Kesten variables) is readjusted by a factor $e^{f(w_t, \{\lambda_t, b_t, ...\})}$ reflecting the constraints imposed on the dynamical process. It is thus reasonable to consider the case where $f(w_t, \{\lambda_t, b_t, ...\})$ depends on t only through the dynamical variables λ_t (and in special cases b_t), a condition which already holds for the two examples above. In the following Fokker-Planck approximation, we shall consider the case where $f(w_t, \{\lambda_t, b_t, ...\})$ is actually a function of the product $\lambda_t w_t$, which is the value generated by the process at step t and to which the constraint represented by $f(\lambda_t w_t)$ is applied. We shall turn back to the general case (24) in Section 2.5.

In the Fokker-Planck approximation, $f(\lambda_t w_t)$ defines an effective repulsive stochastic force. To illustrate the repulsive mechanism, it is enough to consider the restricted case where $f(w_t)$

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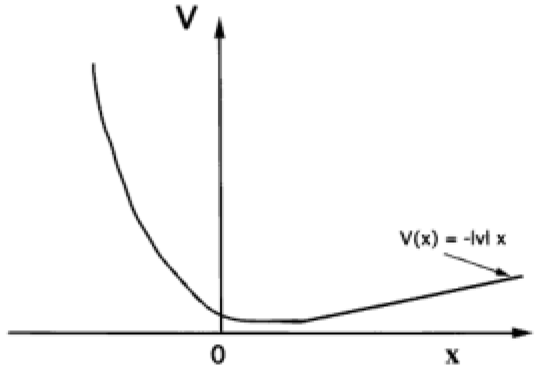

Fig. 4. — Generic form of the potential whose gradient gives the force felt by the random walker. This leads to a steady-state exponential profile of its density probability, corresponding to a power law distribution of the w_t-variable.

is only a function of w_t. This corresponds to freezing the random part in the noise term λ_t leading to the definition of the diffusion coefficient. In the random walk analogy, we thus have the force $F(x_t) = f(w_t)$ acting on the random walker. The corresponding Fokker-Planck equation is

$$\frac{\partial \mathcal{P}(x,t)}{\partial t} = -\frac{\partial (v + F(x))\mathcal{P}(x,t)}{\partial x} + D\frac{\partial^2 \mathcal{P}(x,t)}{\partial x^2}. \tag{25}$$

$F(x)$ decays to zero at $x \to \infty$ and establishes a repulsion of the diffusive process in the negative x region: this is the translation in the random walk analogy of the condition $f(w_t) \to \infty$ for $w_t \to 0$.

With these properties, the tail of $\mathcal{P}(x)$ for large x and large times is given by $\mathcal{P}_\infty(x) \sim e^{-\mu x}$, and as a consequence w_t is distributed according to a power law, with exponent μ given again approximately by (11, 15). The shape of the potential defined by $v + F(x) = -\frac{\partial V(x)}{\partial x}$, showing the fundamental mechanism, is depicted in Figure 4. As we have already noted, the bound w_0 leading to a reflecting barrier is a special case of this general situation, corresponding to a concentrated repulsive force at x_0.

The expression (24) for the general model can be "derived" from the overdamped Langevin equation equivalent to the Fokker-Planck equation (25):

$$\frac{dx}{dt} = F(x) - |v| + \eta(t). \tag{26}$$

Let us take the discrete version of (26) as $x_{t+1} = x_t + F(x_t) - |v| + \eta_t$, replace with $x_t = \log w_t$ and exponentiate to obtain

$$w_{t+1} = e^{F(\log w_t)} \lambda_t w_t, \tag{27}$$

where $\lambda_t \equiv e^{-|v|+\eta_t}$. Since $F(x) \to 0$ for large w_t, we recover a pure multiplicative model $w_{t+1} = \lambda_t w_t$ for the tail. The condition that $F(x)$ becomes very large for negative x ensures that w_t cannot decrease to zero as it gets multiplied by a diverging number when it goes to zero.

2.5. Exact Derivation of the Tail of the Power Law Distribution. — The existence of a limiting distribution for w_t obeying (24), for a large class of $f(w, \{\lambda, b, ...\})$ decaying to zero for large w and going to infinity for $w \to 0$, is ensured by the competition between the convergence of w to zero and the sharp repulsion from it. We shall also suppose in what follows that $\partial f(w, \{\lambda, b, ...\})/\partial x \to 0$ for $w \to \infty$, which is satisfied for a large class of smooth functions already satisfying the above conditions. It is an interesting mathematical problem to establish this result rigorously, for instance by the method used in [1, 10]. Assuming the existence of the asymptotic distribution $P(w)$, we can determine its shape, which must obey

$$v \equiv w\, \mathrm{e}^{-f(w,\{\lambda,b,...\})} \stackrel{\text{law}}{=} \lambda w, \tag{28}$$

where $\{\lambda, b, ...\}$ represents the set of stochastic variables used to define the random process. The expression (28) means that the l.h.s. and r.h.s. have the same distribution. We can thus write

$$P_v(v) = \int_0^{+\infty} \mathrm{d}\lambda\, \Pi(\lambda) \int_0^{+\infty} \mathrm{d}w\, P_w(w)\delta(v - \lambda w) = \int_0^{+\infty} \frac{\mathrm{d}\lambda}{\lambda} \Pi(\lambda) P_w(\frac{v}{\lambda}).$$

Introducing $V = \log v$, $x \equiv \log w$ and $l \equiv \log \lambda$, we get

$$P(V) = \int_{-\infty}^{+\infty} \mathrm{d}l\, \Pi(l) P_x(V - l). \tag{29}$$

Taking the logarithm of (28), we have $V = x - f(x, \{\lambda, b, ...\})$, showing that $V \to x$ for large $x > 0$, since we have assumed that $f(x, \{\lambda, b, ...\}) \to 0$ for large x. We can write $P(V)\mathrm{d}V = P_x(x)\mathrm{d}x$ leading to $P(V) = \frac{P_x(x(V))}{1-\partial f(x,\{\lambda,b,...\})/\partial x} \to P_x(V)$ for $x \to \infty$. We thus recover the Wiener-Hopf integral equation (18) yielding the announced results (16) with (17) and therefore the power law distribution (14) for w_t with μ given by (17).

This derivation explains the origin of the generality of these results to a large class of convergent multiplicative processes repelled from the origin.

3. Discussion

3.1. Nature of the Solution. — To sum up, convergent multiplicative processes repelled from the origin lead to power law distributions for the multiplicative variable w_t itself. Ideally, this holds true in the asymptotic regime, namely after an infinite number of stochastic products have been taken. This addresses a different question than that answered by the log-normal distribution for unconstrained processes which describes the convergence of the reduced variable $\frac{1}{\sqrt{t}}(\log w_t - \langle \log w_t \rangle)$ to the Gaussian law. Notice that this reduced variable tends to zero for our problem and thus does not contain any useful information.

We have presented an intuitive approximate derivation of the power law distribution and its exponent, using the Fokker-Planck formulation in a random walk analogy. Our main result is the explicit calculation of the exponent of the power law distribution, as a solution of a Wiener-Hopf integral equation, showing that it is controlled by extreme values of the process. We have also been able to extend the initial problem to a large class of systems where the common feature is the existence of a mechanism repelling the variable away from zero. We have in particular drawn a connection with the Kesten process well-known to produce power law distributions. The results presented in this paper are of importance for the description of many systems in Nature showing complex intermittent self-similar dynamics.

3.2. The Exponent μ. — In the Fokker-Planck approximation of the random walk analogy, μ is the inverse of the size of the effective cavity trapping the random walk. In this approximation, μ is a function of, and only of, the first two cumulants of the distribution of $\log \lambda$. In particular, if the drift $|v| < 2D$, $\mu < 2$ corresponding to variables with no variance and even no mean when $\mu < 1$ ($|v| < D$). It is rather intuitive: large fluctuations in λ lead to a large diffusion coefficient D and thus to large fluctuations in w_t quantified by a small μ. Recall that the smaller μ is, the wilder are the fluctuations.

Within an exact formulation, we have shown that there is a rather subtle phenomenon which identifies μ as the inverse of the typical value of the largest excursion against the flow of a particle in random motion with drift. This holds true for a large class of models characterized by a negative drift and a sufficiently fast repulsion from the negative domain (in the x-variable), *i.e.* from the origin (in the w-variable).

3.3. Additional Constraint Fixing μ. — We recover the relationship relating μ to the minimum value w_0 in the reflecting barrier problem by specifying [1] the value C of the average $\langle w_t \rangle$. Calculating the average straightforwardly using (14), we get $\langle w_t \rangle = w_0 \frac{\mu}{\mu-1}$, leading to

$$\mu = \frac{1}{1-(w_0/C)}. \tag{30}$$

Notice that this expression is a special case of (17) and should by no mean be interpreted as implying that μ is controlled by w_0 in general. This is only true with an *additional* constraint, here of fixing the average. The *general* result is that μ is given by (17), *i.e.* at a minimum by the two first cumulants of the distribution of $\log \lambda$.

3.4. Positive Drift in the Presence of an Upper Bound. — Consider a purely multiplicative process where the drift is reversed $\langle \log \lambda \rangle > 0$, corresponding to an average exponential growth of w_t in the presence of a barrier w_0 limiting w_t to be *smaller* than it. The same reasoning holds and a parallel derivation yields

$$P_\infty(w_t) = \frac{\mu}{w_0^\mu} w_t^{\mu-1}, \tag{31}$$

with $\mu \geq 0$ again given by (17). This distribution describes the values $0 < w_t < w_0$. Notice that, if $\mu > 1$, the distribution is *increasing* with w_t. This is obviously no more a power law of the tail, rather a power law for the values close to zero. For $\mu < 1$, $P_\infty(w_t)$ decays as a power law, however bounded by w_0 and diverging at zero (while remaining safely normalized). This shows that, when speaking of general power law distribution for large values, this regime is not relevant. Only the regime with negative drift and lower bound is relevant.

However, in the case of Kesten variables (21), if S_t is growing exponentially with an average rate $\langle \log \lambda_t \rangle > 0$, and if the input flow b_t is also increasing with a *larger* rate r, we define $b_t = e^{r(t+1)} \hat{b}_t$, where $\hat{b}_t$ is a stochastic variable of order one. We also define $\lambda_t = \hat{\lambda}_t e^r$. If $r > \langle \log a_t \rangle$, then $\langle \log \hat{\lambda}_t \rangle < 0$.

The equation (1) thus transforms into $\hat{S}_{t+1} = \hat{\lambda}_t \hat{S}_t + \hat{b}_t$, with $S_t = e^{rt} \hat{S}_t$, and where $\hat{\lambda}_t$ and $\hat{b}_t$ obey exactly the conditions for our previous analysis to apply. The conclusion is that, due to input growing exponentially fast, the growth rate of w_t becomes that of the input, its average (which exists for $\mu > 1$) grows exponentially as $\langle S_t \rangle \sim e^{rt}$ and its value exhibits large fluctuations governed by the power law probability density function $P(S_t) \sim \frac{e^{\mu r t}}{S_t^{1+\mu}}$ with μ solution of $\langle \lambda_t^\mu \rangle = e^{r\mu}$, leading to $\mu = \frac{\langle b_t \rangle - \langle \lambda_t \rangle}{\langle \lambda_t^2 \rangle - \langle \lambda_t \rangle^2}$ in the second order cumulant approximation.

3.5. Transient Behavior. — For t large but finite, the exponential (16) with (17) is truncated and decays typically like a Gaussian for $x > \sqrt{Dt}$. Translated in the w_t variable, the power law distribution (14) extends up to $w_t \sim e^{\sqrt{Dt}}$ and transforms into an approximately log-normal law for large values. Refining these results for finite t using the theory of renewal processes is an interesting mathematical problem left for the future.

3.6. Non-Stationary Processes. — When the multiplicative process (1) is not stationary in time, for instance if $v(t), D(t)$ or $x_0(t)$ become function of time, then their characteristic time τ of evolution must be compared with $t^*(x) = x^2/D$. For "small" x such that $t^*(x) \ll \tau$, the distribution $P(x,t)$ keeps an exponential tail with an exponent adiabatically following $v(t), D(t)$ or $x_0(t)$. We thus predict a power law distribution for w_t but with an exponent varying with v and D according to equations (11, 15). For "large" x such that $t^*(x) \geq \tau$, the diffusion process has not time to reach x and to bounce off the barrier that the parameters have already changed. It is important to stress again the physical phenomenon at the origin of the establishment of the exponential profile: the repeated encounters of the diffusing particle with the barrier. For large x, the repeated encounters take a large time, the time to diffuse from x to the barrier back and forth. In this regime $t^*(x) \geq \tau$, the exponential profile for $P(x)$ has not time to establish itself since the parameters of the diffusion evolve faster that the "scattering time" off the barrier. The analysis of the modification of the tail in the presence of non-stationarity effects is left to a separate work. In particular, we would like to understand what are the processes which lead to an exponential cut-off of the power law in the w_t variable, corresponding to an exponential of an exponential cut-off in the x-variable.

3.7. Status of the Problem. — Levy and Solomon [1] propose that the power law (14) is to multiplicative processes what the Boltzmann distribution is to additive processes. In the latter case, the fluctuations can be described by a single parameter, the temperature (β^{-1}) defined from the factor in the Boltzmann distribution $e^{-\beta E}$. In a nutshell, recall that the exponential Boltzmann distribution stems from the fact that the number Ω of microstates constituting a macro-state in an equilibrium system is multiplicative in the number of degrees of freedom while the energy E is additive. This holds true when a system can be partitionned into weakly interactive sub-systems. The only solution of the resulting functional equation $\Omega(E_1 + E_2) = \Omega(E_1)\Omega(E_2)$ is the exponential.

No such principle applies in the multiplicative case. Furthermore, the Boltzmann reasoning that we have used in Section 2.1 is valid only under restrictive hypotheses and provides at best an approximation for the general case. We have shown that the correct exponent μ is in fact controlled by *extreme* excursions of the drifting random walk against the main "flow" and not by its average behavior. This rules out the analogy proposed by Levy and Solomon.

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