Nowak project

1. Linear case:

$x_1' = (1-q)\sum_{n\geq1} a_n x_n - (qa_1 + d_1)x_1$
$x_n' = q\,a_{n-1}x_{n-1} - (qa_n + d_n)\,x_n.$
$y' = by - dy.$

Eigenvalue condition for the x equation:

$$1 = \frac{1-q}{q}\sum_{n\geq1}\prod_{k=1}^{n}\frac{qa_k}{(\lambda + (qa_k + d_k))} \tag{1.1}$$

Note that $\lambda > 0$ requires that

$$\frac{1-q}{q}\sum_{n\geq1}\prod_{k=1}^{n}\frac{qa_k}{(qa_k + d_k)} \geq 1 \tag{1.2}$$

The condition $\lambda > b - d$ is needed for growth faster than that of y. This condition reads

$$\frac{1-q}{q}\sum_{n\geq1}\prod_{k=1}^{n}\frac{qa_k}{(b-d + (qa_k + d_k))} > 1\ . \tag{1.3}$$

In the case when $a_k = a$ and $d_k = d$ is constant, then the condition in (1.1) asserts that $1 = \frac{1-q}{q}\sum_{n\geq1}\eta^n$ with $\eta = qa(\lambda + qa + d)^{-1}$. This is to say that $\frac{q}{1-q} = \eta/(1-\eta)$ and so $\eta = q$. Thus, $\lambda + qa + d = 2qa$ and so $\lambda = (1-q)a - d$. Growth faster than the y-model requires $(1-q)a > b$ which is maybe expected.

Martins 'system with food' on page 2 at equilibrium $z^* = d/b$ gives the linear instability condition that is identical to (1.2) with the replacement $q \to z^*q$. This understood, I will address the remaining questions on the bottom of page 2 with $z^* = 1$.

a) Neutrality

Martin suggests considering the case $d_k = d$ in which case the condition $\lambda = b - d$ reads

$$\frac{1-q}{q}\sum_{n\geq1}\prod_{k=1}^{n}\frac{qa_k}{(qa_k + b)} = 1\ . \tag{1.4}$$

Martin claims that this condition is obeyed if $a_k = kb$. In the latter case, the condition in (1.4) reads

$$\frac{1-q}{q}\sum_{n\geq 1}\prod_{k=1}^{n}\frac{qk}{(qk+1)} = 1\ .$$

(1.5)

To verify that this is indeed the case, introduce for the moment η to denote 1/q. What is written in (1.5) is equivalent to the assertion that

$$\sum_{n\geq 1}\prod_{k=1}^{n}\frac{k}{(k+\eta)} = \frac{1}{\eta - 1}\ .$$

(1.6)

A given term in this sum is equal to

$$\eta\int_0^\infty \frac{t^n}{(1+t)^{n+1+\eta}}\,dt\ .$$

(1.7)

as can be seen using n successive integration by parts. This being the case, interchange the sum and the integral. The result on the left side of (1.6) is then

$$\eta\int_0^\infty \frac{t}{(1+t)^{2+\eta}}\sum_{n\geq 0}\left(\frac{t}{1+t}\right)^n dt\ .$$

(1.8)

The sum in the integrand is geometric, and what is written above is equal

$$\eta\int_0^\infty \frac{t}{(1+t)^{1+\eta}}\,dt = \int_0^\infty \frac{1}{(1+t)^{\eta}}\,dt\ .$$

(1.9)

The right hand integral is indeed equal to $\frac{1}{\eta - 1}$.

b) $a_k = b$ for $k < m$ and $a_k = a$ for $k > m$

Martin asks for the case $a_k = b$ for $_k < m$ and $a_k = a$ for $_k \geq m$ with $a > b$. I assume again that all $d_k = d$. In this case, the left hand side of (1.4) reads

$$\frac{1-q}{q}\ \sum_{1\leq n<m}\left(\frac{q}{(q+1)}\right)^n + \left(\frac{qa}{(qa+b)}\right)^m\sum_{k\geq 0}\left(\frac{qa}{(qa+b)}\right)^k\ ,$$

(1.10)

Evaluating these sums gives the instability condition

$$\left(\frac{qa}{(qa+b)}\right)^{m-1}\frac{a}{b} - \left(\frac{q}{(q+1)}\right)^{m-1} > \frac{q}{1-q}\,.$$

(1.11)

c) a_k is a rational function of k

The next case Martin asks about is that where $a_k = (c_0k - c_1)/(k + c_2)$ where the constants are chose so that $b = (c_0 - c_1)/(1 + c_2)$. The neutrality condtion in (1.4) reads

$$\frac{1-q}{q} \sum_{n\geq 1} \prod_{k=1}^{n} \frac{q(c_0 k - c_1)}{(qc_0+b)k + (bc_2 - qc_1)} = 1 \ . \tag{1.12}$$

This can be rewritten as

$$\frac{1-q}{q} \sum_{n\geq 1} \gamma^n \prod_{k=1}^{n} \frac{k-\alpha}{k+\beta} = 1 \ , \tag{1.13}$$

where $\gamma = \frac{qc_0}{qc_0+b}$, $\alpha = \frac{c_1}{qc_0}$, and $\beta = \frac{bc_2 - qc_1}{qc_0 + b}$. The n'th term in the sum in (1.13) can be written as

$$\rho^{-1}\int_0^\infty \frac{t^{n-\alpha}}{(1+t)^{n+1+\beta}}\,dt \quad \textit{where} \quad \rho = \int_0^\infty \frac{t^{-\alpha}}{(1+t)^{1+\beta}}\,dt \ . \tag{1.14}$$

This understood, interchange the integral with the sum to rewrite the sum in (1.13) as

$$\gamma\rho^{-1}\int_0^\infty \frac{t^{-\alpha}}{(1+t)^{1+\beta}} \sum_{n\geq 0} \left(\frac{\gamma t}{1+t}\right)^n dt \ = \ \gamma\,\rho^{-1}\int_0^\infty \frac{t^{-\alpha}}{(1+t)^{\beta}(1+(1-\gamma)t)}\,dt \ . \tag{1.15}$$

The stability condition in (1.12) can be restated as

$$\frac{qc_0}{qc_0+b}\int_0^\infty \frac{t^{-\alpha}}{(1+t)^{\beta}(1+(1-\gamma)t)}\,dt \ > \ \frac{q}{1-q}\int_0^\infty \frac{t^{-\alpha}}{(1+t)^{1+\beta}}\,dt \ . \tag{1.16}$$

According to Gradshteyn and Ryzhik, (*Tables of integrals, series and products; Enlarged edition*, I. S. Gradshteyn and I. M. Rhyzik; Academic Press 1980), these definite integrals can be expressed in terms of two special functions, these denoted by B (this being the 'beta function' or 'Euler's integral of the first kind') and F (this being 'Gauss' hypergeometric function'). In particular, Equation 9 in Section 3.197 writes

- $\int_0^\infty \frac{t^{-\alpha}}{(1+t)^{\beta}(1+(1-\gamma)t)}\,dt \ = (1-\gamma)^{\alpha+\beta-1}\, B(\alpha+\beta, 1-\alpha)\ F(\beta, \alpha+\beta\,; 1+\beta; \gamma).$
- $\int_0^\infty \frac{t^{-\alpha}}{(1+t)^{1+\beta}}\,dt = B(\alpha+\beta, 1-\alpha)\ F(\beta, \alpha+\beta\,; 1+\beta; 0)\ .$

(1.17)

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For what it is worth, the special functions B and F are defined respectively in Sections 8.38 and 9.10-13 of Gradshteyn and Ryzhik.

d) Interpreting the instability condition

Martin asks for the meaning of the condition that

$$\frac{1-q}{q} \sum_{n \geq 1} \prod_{k=1}^{n} \frac{qa_k}{(qa_k + b)} > 1. \tag{1.18}$$

Setting $\alpha_k = \frac{qa_k}{(qa_k + b)}$, this is equivalent to the condition that

$$\alpha_1 + \alpha_1\alpha_2 + \alpha_1\alpha_2\alpha_3 + \cdots > \frac{q}{1-q} . \tag{1.19}$$

What follows is a thought about an interpretation: Looking at the equation for $x_{k>1}$, I can think of α_{k-1} as the probability of creating some x_k given x_{k-1}. This understood, α_1 is the probability of having x_2 given x_1, then $\alpha_1\alpha_2$ is the probability of x_3 given x_1 and $\alpha_1\alpha_2\alpha_3$ is the probability of x_4 given x_1, etc. The sum on the right can be thought of as a sum of conditional probabilities.

I shall think more about this as a path to an interpretation of (1.19).

e) Other forms of density regulation

I haven't had time to consider these yet.